DEPARTMENT OF APPLIED MATHEMATICS,
UNIVERSITY COLLEGE, UNIVERSITY OF LONDON.

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TECHNICAL SERIES, IV.

ON TORSIONAL VIBRATIONS IN AXLES AND SHAFTING.

BY

KARL PEARSON, F.R.S.

PROFESSOR OF APPLIED MATHEMATICS AND MECHANICS, UNIVERSITY COLLEGE, LONDON.

[WITH THREE FIGURES IN THE TEXT AND THIRTEEN LITHOGRAPHED TABLES AND PLATES]

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K. P.

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K. P.

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On Torsional Vibrations in Axles and Shafting.

By Karl Pearson, F.R.S.

1. The object of the following paper is to deal with certain cases of impulsive torsion which seem of considerable practical importance, but which so far as I am aware have not hitherto been considered theoretically. I deal in particular with shafts and axles. The following assumptions are made:

(*a*) The proportionality of stress and strain is supposed to hold for the material of the shafting. The limits of safe impulsive loading given in the paper are calculated on the basis of stress remaining proportional to strain up to that limit. The material must be considered then as having been brought to a state of ease both for positive and negative slide, and that between the limits of this state of ease the generalised Hooke's Law holds. This limit to the state of ease is properly the practical efficiency of the material, and experiments on its determination for a given material subjected to a given process of working and afterwards tested to a given load ought to replace the old habit of considering absolute strength and using a large factor of safety.

(*b*) The dimensions of the cross-section treated as uniform are supposed small as compared with the length of the shaft. At the same time by the use of de Saint-Venant's torsion factors it is not necessary to suppose that the cross-sections remain plane. Our results also will be approximately true even when the dimensions of the cross-sections are not extremely small with regard to the length of the shaft. The cross-section may have any form whatever.

(*c*) The material will be supposed to have a uniform slide modulus μ across the cross-section, otherwise the distribution of elasticity may be aeolotropic. In the case of metal cast or drawn, there is little doubt that μ would be different for the core and the periphery; at the same time the core is not that part which contributes the greater portion of the resistance to torsion, while in other cases the epidermis would be turned off. For practical purposes no great error is likely to be introduced by treating μ as uniform for all elements of the cross-section.

(*d*) I suppose the shafting subject only to torsional stress; the conditions for the safe load when in addition to torsional load there is a tractive, flexural, or shearing load, must be obtained by the methods indicated by de Saint-Venant in the Memoir on Torsion, Chapitre XIII. (*Savants étrangers* XIV. pp. 522—558) or in *Leçons de Navier* (pp. 372—500).

2. Let x be the distance measured from one end of the prism along its central axis up to any cross-section (area ω); let ψ be the angle through which the section has been revolved owing to torsion round the central axis, then for a cross-section of any size the corresponding moment of torsion is given by

$$M=\mu\nu\frac{d\psi}{dx} \text{ ....(i)},$$

where ν is de Saint-Venant's torsion factor.

If k be the swing radius of the cross-section about the central axis and it *is allowable to apply* (i) *to the case of vibrations,* we obtain for the case of torsional vibrations the equation

$$\rho\omega k^2\frac{d^2\psi}{dt^2}=\frac{dM}{dx}=\mu\nu\frac{d^2\psi}{dx^2},$$

where t is the time and ρ the density. To obtain this equation we have to assume that the statical torsional couple M will be produced in each element of the shafting. We shall write this equation

$$\frac{d^2\psi}{dt^2}=J^2\frac{d^2\psi}{dx^2} \text{ ....(ii)},$$

where $$J^2=\frac{\mu\nu}{\rho\omega k^2} \text{ ....(iii)},$$

or J represents the velocity of a wave of torsional vibration along the given shafting (*History of Elasticity,* Vol. I. Art. 1629 or *Comptes Rendus,* T. 28, pp. 69—72).

3. The value of the torsion-factor ν has been ascertained by de Saint-Venant for a wide range of cases in the above cited Memoir on Torsion and in the *Comptes Rendus* T. 87 (pp. 893 et seq.). In the *Comptes Rendus* T. 88, pp. 142—7, he has shewn that for practical purposes we may represent it by the empirical formula

$$\nu=\frac{1}{40}\cdot\frac{\omega^4}{\omega k^2}\text{.......................................(iv)}.$$

4. Let σ be the slide, S the shear at any point of the cross-section. Then $S=\mu\sigma$. de Saint-Venant in the memoirs referred to has calculated the value of σ in terms of $\tau\,(=d\theta/dx)$ the angle of torsion, for any point of a great variety of cross-sections.

Let S_0 be the value of S at which σ passes the limit to the state of ease, or the material for practical purposes 'fails.' I shall call this the *fail-limit* of the material, and the point in the cross-section at which it occurs the *fail-point.* If we know S and M in terms of the angle of torsion, we can find by elimination a relation between them, we thus obtain a value for the maximum safe couple M_0 in terms of S_0 in the form

$$M_0=\gamma' S_0\sqrt{\omega^3}\text{.......................................(v)},$$

γ' being a coefficient depending on the form of the cross-section. de Saint-Venant

(*Leçons de Navier*, p. 366) tabulates the values of γ' for a considerable number of cross-sections. Thus for:

Circular Section	$\gamma' = \cdot 2821$,
Square Section	$\gamma' = \cdot 2082$,
Square Section with curved sides and rounded angles	$\gamma' = \cdot 2001$,
Square Section with curved sides and acute angles	$\gamma' = \cdot 1919$,
Equilateral Triangular Section	$\gamma' = \cdot 1755$,
Star with four points	$\gamma' = \cdot 1607$,

while he also gives the value of γ' for elliptic and rectangular sections of which the diameters have a ratio varying from 1 to 8.

For practical purposes we may take for:

Elliptic Section	$\gamma' = \cdot 2821\sqrt{n}$.
Rectangular Section	$\gamma' = \frac{1}{3} \cdot \frac{5}{3n + 5/n}$,

where n is the ratio of the lesser to the greater diameter of the section, *i.e.* $n < 1$: see the Memoir on Torsion pp. 351—414*. The former result being accurate to four decimal places, the latter empirical and only approximate.

For a hollow shaft of which the thickness $= n$ times the outer radius

$$\gamma' = \cdot 2821 \frac{2 - 2n + n^2}{\sqrt{2n - n^2}}, \quad = \cdot 2821 \left\{\sqrt{2}\left(\frac{1}{\sqrt{n}} - \sqrt{n}\right)\right\} \text{ if } n \text{ be small.}$$

For such shafts there is obviously a great gain in strength.

The fail-points in all these cases are the points nearest to the central axis of the shaft. As a rule the greater portion of a shaft will be of circular or hollow circular section, but in some cases the cross-section near the terminals will be given a different form for the purpose of a joint or fitting. In such a case the square shape will be better than any given above except the ellipse, but by referring to de Saint-Venant's tables we see that an elliptic section for which the ratio of the axes is even so great as 1·8, will give a stronger fitting than a square section of equal area in the ratio of ·2103 to ·2082, while one in which the ratio of the axes is 1·2, will be stronger in the ratio of ·2575 to ·2082.

5. The values of μ the slide modulus will vary with the material out of which the shaft is constructed. Where no better data are available μ may be taken $\frac{2}{5}$ of E, the stretch modulus, the latter being more generally known or at least more readily ascertainable.

In order to keep before us some idea of the numerical magnitude of the quantities under consideration, I will calculate roughly the value of J the velocity of a wave of torsional vibration in a wrought iron shaft of circular cross-section. In this case we may take μ about 10,500,000 lbs. per sq. in. I take this value from my reduction of Kuppfer's experiments for iron, No. 1: see *History of Elasticity*, Vol. I. Art. 1394. For a circular cross-section $\nu = \omega k^2$, and for wrought iron we may take the cubic foot to weigh 480 lbs. Then we have

---

* I do not understand on what ground Cotterill bases the formula given in the Table, p. 361 of his Applied Mechanics.

$$J = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{10{,}500{,}000 \times 144 \times g}{480}}$$

$$= 10{,}090 \text{ feet per sec. about*.}$$

Thus the vibration, or the torsional couple will be transferred along a foot of wrought iron shafting in about $\frac{1}{10000}$ of a second. In the particular case of a simple wave in the shafting a wave of length λ will repeat itself every $\lambda/10{,}000$ secs. or if l represents its length every $\frac{l}{10{,}000} \times \frac{\lambda}{l}$ secs. For such a case with the various terminal and other conditions of practice, λ may be 40 or 50 or more times l, hence, roughly speaking, the wave would repeat itself only 5 times a second, or this is the number of times the maximum stresses recur in a second.

6. *Case* (*i*). *To one end of a shaft of l feet length a couple N is suddenly applied. The other end of the shaft is either built in, or else it needs a greater couple than N to overcome the frictional resistance applied to a wheel or other mechanical arrangement at that terminal. It is required to consider the stress at any time after the couple is impulsively applied.*

Let $\mu\nu = K$. Then the couple N we will suppose applied at $x = l$; at $x = 0$, $\psi = 0$ always. A suitable solution of the equation (ii) is

$$\psi = \Sigma A_m \{\cos(Jmt)\} \{\sin(mx)\} + \frac{N}{K} x,$$

which gives
$$M = K\frac{d\psi}{dx}, \quad = N \text{ if } x = l$$

and
$$ml = \frac{\pi}{2}(2r+1).$$

Obviously $\frac{d\psi}{dt} = 0$ for all values of x when $t = 0$, and ψ will equal nothing for all values of x, when $t = 0$, if

$$0 = \sum_{r=0}^{r=\infty} A_m \sin mx + \frac{N}{K} x.$$

Thus A_m is easily determined by Fourier's Theorem, and we have

$$\psi = \frac{N}{R}\left\{x - \frac{8l}{\pi^2}\sum_0^\infty \frac{(-1)^r}{(2r+1)^2} \cos J\frac{\pi}{2l}(2r+1)\,t \sin\frac{\pi}{2l}(2r+1)\,x\right\}.$$

---

* For the practical numerical calculations of this paper it suffices to take $J = 10{,}000$.

The series of trigonometrical products admits of being broken up into two, which allow of discontinuous summation, namely if ϕ lies between $\pm\frac{\pi}{2}$, we know that

$$\frac{\pi\phi}{4}=\sin\phi-\frac{1}{3^2}\sin 3\phi+\frac{1}{5^2}\sin 5\phi-\frac{1}{7^2}\sin 7\phi+\dots$$

Hence we can easily deduce that the expression on the right-hand side is equal to

$$(-1)^r\frac{\pi}{4}(\phi-r\pi) \text{ if } \phi \text{ lies between } \overline{2r-1}\,\frac{\pi}{2} \text{ and } \overline{2r+1}\,\frac{\pi}{2}.$$

Since $\cos Jt\frac{\pi}{2l}\sin\frac{x\pi}{2l}=\frac{1}{2}\left\{\sin(x+Jt)\frac{\pi}{2l}+\sin(x-Jt)\frac{\pi}{2l}\right\}$ we have the following results:

(*a*) $(Jt+x)<l$, $\psi=0$;

(*b*) $(Jt+x)>l,\ <l+x$, $\psi=\frac{N}{K}(x-l+Jt)$;

(*c*) $Jt>(l+x),\ <(3l-x)$, $\psi=\frac{N}{K}(2x)$;

(*d*) $Jt>(3l-x),\ <(3l+x)$, $\psi=\frac{N}{K}(x+3l-Jt)$;

(*e*) $Jt>(3l+x),\ <(5l-x)$, $\psi=0$;

(*f*) $Jt>(5l-x),\ <(5l+x)$, $\psi=\frac{N}{K}\{x-5l+Jt\}$;

(*g*) $Jt>(5l+x),\ <(7l-x)$, $\psi=\frac{N}{K}(2x)$;

(*h*) $Jt>(7l-x),\ <(7l+x)$, $\psi=\frac{N}{K}\{x+7l-Jt\}$.

etc. etc.

There is no difficulty now in writing down the general value of ψ for any value of t. It depends entirely on the number of times the 'head of the wave' has passed and repassed on reflection at the built-in end the section at which we require ψ.

Results similar to the above (*a*)—(*h*) would have been reached even more readily had we adopted the method of solution by arbitrary functions for Equation (ii).

The Figures I (*a*) to I (*c*) on Plate I give the values after any time at any point of the shafting of the angle ψ, the torsional couple $M=K\,d\psi/dx$, and the slide-speed $d\psi/dt$. Distances along the vertical give the length of any section of the shafting from the built-in end, the time as given by Jt is plotted off along the horizontal.

These figures lead to some interesting conclusions; (α) the displacement from section to section is continuous and varies from 0 to $(N/K)\,2x$. (β) The torsional couple is discontinuous and takes only three (all positive) values, namely 0, N and $2N$.

We note that the displacement and the couple will take as maxima-values *double* what they would have done at each point under a couple statically applied. This is of course only a special case of Young's General Theorem. (γ) The displacement speed is discontinuous, and can only take three values, 0, $(N/K)\,J$, and $-(N/K)\,J$, one of which we note is negative.

Like the torsional couple it has, however, instantaneously only two values, one for either element into which the shaft is divided by the alternate 45° sloping lines.

The practical rule as to strength in this Case (i) is evidently the general rule of all impulsive stress in elastic material: The impulsive couple must not be greater than a moiety of the statical couple which would produce a failure of the material. With regard to this rule we note that the maximum couple $2N$ lasts longest at the built-in section; here it lasts for a time $= 2l/J$, that is while the wave of torsional vibration travels backwards and forwards along the shaft. In the case of a wrought iron circular shaft of 5 ft. length (see Art. 5) this is about 1/1000 of a second. It is possible that the torsional couple lasting only this time would not injure the shaft like a statical torsional couple would do when applied for an indefinite time. We note that for the next 1/1000th second the built-in end suffers no stress, after which time the couple $= 2N$ is repeated for 1/1000th second and so on alternately.

It seems to me that not improbably this repetition of the stress, although of course to a smaller degree of magnitude as the vibrations subside, will produce nearly the same injury to the shaft as if there were a statical load upon it equal to $2N$ for half the length of time. We may note further that if the resistance to be overcome lie between N and $2N$, then although the gradually applied (statically applied) couple N will not suffice to overcome it, yet the impulsive couple N will do so. Examples of this principle are common enough, as in turning a screw into a hard piece of wood, or breaking by twisting a stiff piece of wire—a sudden twist will often be effectual.

7. *Case (ii). After τ seconds have elapsed since the application of the impulsive couple N, let a second impulsive couple N' be applied. To draw a diagram of the wave of torsional couple.*

All we have to do in this case is to shift Fig. I. Plate I. horizontally through a distance $J\tau$, replace N by N' and add the quantities in each compartment. The result will depend on where $J\tau$ lies between 0 and $4l$, $4l/J$ being the time the wave takes to pass twice backwards and forwards along the shaft, or the whole time of a complete set of stress-changes.

The typical cases are the following:

(*a*) $J\tau$ lies between 0 and $2l$, or N' is applied before the wave has returned to loaded section of shaft.

(*b*) $J\tau = 2l$, or load N' is applied at the instant the wave has returned to the end of shaft.

(*c*) $J\tau$ lies between $2l$ and $4l$, or N' is applied between first and second reflections of wave from loaded end of shaft.

(*d*) $J\tau = 4l$, or load N' is applied on second reflection of wave from loaded end of shaft. This case corresponds exactly to an impulsive couple $N+N'$ being applied after the first complete period.

The couples in these four cases are exhibited in Plate I. Fig. II. (*a*) to (*b*).

These diagrams contain some interesting results. We note again that double of the total load couple $N+N'$ is the maximum stress couple in the first and third cases; we see further that this maximum stress couple is not imposed at each section of the rod during the vibrations. In other words the danger of failure is not as in the case of statical torsion the same for every cross-section. Thus in the case of (*a*), only sections between $x=0$, and $x=l-J\tau/2$ are subjected to the maximum couple $2(N+N')$, in the case of (*c*) only sections between $x=0$, and $x=J\tau/2-l$. In addition, in both cases the time during which the stress lasts at any of these sections varies directly as their distance from the section $x=l \sim J\tau/2$. Thus the built-in end of the shaft sustains the maximum couple for the longest time and *ceteris paribus*, we may expect therefore failure to take place at the built-in section. If the built-in end of the shaft be strengthened by a boss or other mass of metal, the section immediately beyond this will be that of greatest danger.

In case (*b*), the maximum couple is only the greater of the two, $2N$ or $2N'$, but we note that after the application of the second impulsive couple, the built-in end of the shaft is never free from stress. A singular result is obtained by putting $N=N'$, in this case after a time $\tau = 2l/J$ the couple remains uniform throughout the shaft and equal to its statical magnitude. Referring to Fig. I. (*c*) of Plate I., we note that if it be superposed on itself after shifting horizontally through a distance $2l$, then $d\psi/dt=0$, for all values of t after $t=2l/J$. Thus we have the following proposition:

If an impulsive couple be applied to a length of shafting, and if at the instant the wave of vibrations returns to the point of application after reflection at a fixed end, a second equal couple be applied, then the whole shaft is brought to vibrational rest, each section being subjected to the statical torsional strain $\left\{\psi = \frac{2N}{K}(x-l)\right\}$.

Of case (*d*), there is little to be noted, except that after the application of N' the shaft vibrates as it would do under an impulsive couple $N+N'$.

8. *Case* (*iii*). *A shaft with one end free and the other end fixed* (*couple not great enough to overcome resistance*) *is subjected to an impulsive couple N, which after τ seconds is removed. To find the stress in the shaft.*

All we have to do is to make $N' = -N$ in the diagrams Fig. II. (a) to (d) of Plate I. We obtain the results Fig. III. (a) to (d) of Plate I., corresponding to the four possible cases of the duration of the load.

In each case the shaded part of the strip marks the interval during which the couple N is applied.

These diagrams suggest some rather important practical conclusions.

(i) Unless the load be removed exactly after a time $4l/J$ (when the shaft remains unstrained), the whole shaft (b), or certain portions of it (a and c), will be subjected alternately to positive and negative couples equal to $\pm 2N$. In the case of an impulsive couple not removed there is only a positive stress couple $= 2N$. Thus the range of stress when the load is suddenly applied and suddenly removed is *double* the range when the load is not removed.

Now certain experiments of Wöhler (see his work, *Ueber die Festigkeitsversuche*) seem to shew that the range of stress must never be greater than the maximum statical safe-stress. *If this law be true* we have the following results:

For statical torsion the load-couple must never be greater than M_0.

When the load-couple is suddenly applied but not suddenly removed, it must not be greater than $M_0/2$.

When the load-couple is suddenly applied and also suddenly removed, it must not be greater than $M_0/4$.

It is possible, however, that the results of the very careful experiments upon which this law of the 'range of stress' is based have been misinterpreted owing to a not very clear understanding of the effect of the time element brought out by the diagrams above. I shall return to this point later.

(ii) In Fig. III. (a), (b), (c) there are times at which the stress-couple vanishes at every point of the shaft with the possible exception of a single point of discontinuity. The times at which this singular distribution of stress arises are as follows: In case (a) and (c) when $t = 2l/J + \tau/2$. In case (b) when $t = 3l/J$. In order to ascertain a little more clearly what takes place, we must return to Fig. I. (a) to (c) of Plate I., we must replace t by $t - \tau$, and N by $-N$, then shifting the figures horizontally through $J\tau$, superposing them on the original diagrams and adding the quantities in the same spaces we obtain the following results:

Case	Time	Section of Discontinuity	Portion of Shaft next	Value of ψ	Value of $d\psi/dt$
(*a*)	$2l/J+\tau/2$	$x=l-J\tau/2$	loaded section built-in section	0 0	$-2JN/K$ 0
(*b*)	$3l/J$	built-in section	whole shaft	0	$-2JN/K$
(*c*)	$2l/J+\tau/2$	$x=J\tau/2-l$	loaded section built-in section	0 0	$-2JN/K$ 0

In case (*d*), $d\psi/dt$ as well as ψ vanishes for all values of x when $t=4l/J$.

Thus we see that although the torsional angle ψ and the stress-couple are zero, yet the slide velocity $-d\psi/dt$ through the whole or a portion of the shaft is not so, so that immediately stresses are again brought into play.

9. We are now in a position to deal with the important case of *repeated loading.*

Case (iv). A shaft built in at one end and free at the other is subjected to a torsional couple N for τ seconds. This couple is again imposed for τ seconds after τ' seconds have elapsed. The process of loading and unloading being repeated every $\tau+\tau'$ seconds, it is required to find the nature of the stress produced.

It is well known that a wave of vibration set up in an elastic material does not continue to move backwards and forwards across the piece indefinitely. The energy of vibration rapidly becomes less and less owing to the viscosity of the material, and within a very short time it is practically at rest. Hence the diagrams such as Fig. III. (*a*) to (*d*) above are only theoretical representations; the successive maxima of stress-couples get less and less, till within a comparatively short time after the removal of the load the maximum stress-couple produced by the motion of the wave is a negligible quantity. Let t_0 be the time it takes for a wave of torsional vibration to be dissipated. Then the solution of the problem of repeated load will depend upon a determination of t_0, and its relation to the duration of loading τ and the interval of no loading τ'. If $\tau'>t_0$ then repeated loading will have no more effect than a single loading, if $\tau'<t_0$ then the maximum couples produced by different loads may occur at the same section at the same instant, and in this case the repeated load may produce a very disastrous effect; the only case in which τ' whether $>$ or $<t_0$, will have no importance, is that of case (*d*), when the time of loading $\tau=4l/J$; under these circumstances repeated loadings, whatever the rapidity of their succession, will have no more effect than a single loading. This case in which the period of loading is equal to, or an exact multiple of, the wave period, is very unlikely to occur in practice, and as it possesses no features of special interest may be dismissed without further consideration.

In experiments on repeated loading I do not think that the importance of noting the exact relations between τ, τ' and t_0 has always been recognised. It is obvious that the quantity t_0 is an essential feature in the working strength of a material subject to rapid and discontinuous changes of load; such for example as must frequently occur in shafting driving machines, which have to overcome discontinuous resistance, or possibly in the impulsive couple produced in the axles of railway waggons by passing the junction of two lengths of rail. I shall return later to such cases of revolving shafting. Experiments to determine the absolute value of t_0 would be impossible, but there would be little difficulty in determining how many times the wave runs backwards and forwards along the beam after the removal of the load before the maximum couples become negligible*. Examining Fig. I. (*a*) and (*b*) of Plate I. we see that this is practically the same problem as determining the manner in which the amplitudes of $\psi\left(=\frac{N}{K}2x \text{ or } \frac{N}{K}2l \text{ at the free end}\right)$ diminish.

In shafting of fair section such as would be used in machinery the terminal conditions are generally such that the distance between maxima of the stress-couples which may be taken to correspond to the wave-length is many multiples of the length of the shaft, hence a considerable interval elapses between the maxima, and viscosity has more opportunity of influencing the maximum than if as in the case of Art. (6) the wave-length is only four times the length of the shaft. Thus in such cases we shall be within the bounds of safety if we take the greatest of the first two or three maxima. It is possible that the effect of viscosity is increased when the area of the cross-section is increased, but I know of no experimental data on the decrease of amplitude of the torsional vibrations of a cylinder with a diameter of several inches. In default of any evidence as to the magnitude of t_0, we must content ourselves with giving it an arbitrary value and then take examples based upon this value for repeated loading. If we do not take any account of viscosity it is obvious that any shaft might be destroyed by repeating even a very small load-couple at very long intervals a sufficient number of times.

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* Professor A. B. W. Kennedy kindly made experiments for me in 1889 on rods of fair size (*e.g.* $\frac{3}{8}''$ diameter and $50''$ long). The oscillations (and consequently the stresses) due to an impulsive couple were of sensible relative magnitude for 18 to 20 secs. after the removal of the load. The reduction was, however, such that I think we should have to look for the *maximum maximorum* among the first two or three *maxima*. In the case of rotating shafting and axles subject to impulsive stress, the circumstances are such as to make the wave-length a very much greater multiple of the shaft-length and hence to increase the effect of viscosity between the maxima. What we need are experiments on axles and shafts such as occur in practice rather than on the wires and rods of the physical laboratory. Researches on repeated loading require to be undertaken with a full appreciation of the importance of the time interval factor; this can hardly be said of Wöhler's experiments.

In Wöhler's experiments of 1870* no data as to the time element are given, but as in some cases he gave upwards of 43,000,000 loadings, it is obvious that an experiment of this kind would have lasted longer than a year†, had the loading been made only once in a second. But it is not probable that for the purpose of his experiments he kept his steam-engines running day and night for a length of time like this. We may I think assume *that Wöhler's loads were repeated in all cases before the stress-wave set up by the previous loading had ceased to be of importance.* Indeed it seems highly probable that several successive loadings were producing effect at one and the same time; hence I am inclined to doubt the generality of Wöhler's Laws and to believe they will ultimately be found to be only a particular case of more general principles involving the consideration of the time element as an important factor.

I shall turn now to special cases which will indicate the general principles referred to above :—

10. Turning to the diagrams Fig. III. (*a*) to (*c*) of Plate I., we note that the maximum stress has its longest duration at the built-in end and has the two values $2N$ and $-2N$ of opposite sign; these may {(*a*) and (*c*)}, or may not {(*b*)} be separated by intervals of zero stress. It will be sufficient then to consider the stress at the built-in end and investigate when the maximum stress will accumulate owing to repeated loadings. We have only to shift the base line of any one of the diagrams through a space $Jt' = J(\tau + \tau')$ to the right to get the effect of a second loading.

(*a*) We will deal first with only a double loading.

Now in order that an accumulative effect may arise, the first $2N$ of the first loading must fall when shifted through Jt', on the second $2N$ of the unshifted loading, or we must have

$$\begin{array}{llll} (a) & Jt' > 4l - J\tau & < 4l + J\tau, & \text{if } J\tau < 2l \\ (b) & Jt' > 2l & < 6l, & \text{if } J\tau = 2l \\ (c) & Jt' > J\tau & < 6l, & \text{if } J\tau > 2l < 4l. \end{array}$$

Let m be the fraction of a second the wave takes to run in one direction along the bar, then $m = l/J$, and we have the following conditions for accumulative effect:

$$\text{(i)}\ldots\ldots\ldots\ldots\ldots\begin{cases} t' \text{ lies between } 4m \pm \tau, & \text{if } \tau < \text{or} = 2m \\ t' \quad ,, \qquad ,, \qquad \tau \text{ and } 6m, & \text{if } \tau > 2m \text{ and } < 4m. \end{cases}$$

It is further necessary that t' be $< t_0$, the time for relative subsidence of stress-wave, or this period must always be greater than $4m - \tau$. If for example $\tau = 2m$ then it is necessary for t_0 to be $> 2m$ or the wave to run without diminished effect at least once backwards and forwards along the bar. If τ be small as compared with m, it is necessary for the wave to run at least twice backwards and forwards for the second loading to accumulate. I imagine that this condition for t_0 will be as a rule fulfilled, and that it is the interval t' between the commencement of successive loadings which practically determines the question of accumulation in the case of two loadings only.

---

* *Ueber die Festigkeitsversuche mit Eisen und Stahl*, Berlin, 1870.

† One experiment with 132,000,000 loadings is said to have lasted 3 years.

We may then conclude that:

If t' satisfies the relation (i) the maximum stress in a twice repeated loading is $4N$, *i.e.* double that of a single impulsive loading, four times that of a gradually applied statical load.

Were Wöhler's law correct as to range of stress we should have a range from $4N$ to $-4N$ and the safe load would be only *one-eighth* of a statical safe load. This I think is very doubtful.

(*b*) Supposing now the load is applied a third time, then these maximum stresses will accumulate provided the condition (i) {modified in case (*c*) to $Jt' > J\tau < 8l - J\tau$} is satisfied and in addition (ii).........$t_0 > 2t'$.

This connotes, if τ be small compared with m, four runnings of the wave backwards and forwards along the shaft without its intensity becoming negligible, and if $\tau = 2m$, two similar movements of the wave. The safe stress limit now becomes 1/6th of the statical load on our theory and 1/12th on Wöhler's theory. *This great reduction of safety is nowhere manifested in Wöhler's experiments.* Hence we are forced to the conclusion that the times at which Wöhler repeated his load and unload did not lead to an accumulation of stress of this overwhelming character.

It is of course obvious that except in the case of an extremely long shaft, we have assumed durations for the load and unload of such smallness that they are hardly likely to occur in practice,—t' as a rule will be many multiples of m.

The general condition will be that t' should lie between $4p \times m \pm \tau$, supposing the time τ to be still very small and p any integer. But these special cases are sufficient to bring out how the maximum stress is dependent on the durations of load and unload and so the incompleteness of Wöhler's laws which do not contain these durations as a necessary element. In my own mind there is little doubt that these durations in Wöhler's experiments were such as to lead to no accumulation of stress due to the first and second loadings. Of course it is possible that the first and third (or a later load) may have done, but it seems idle to investigate the question without some details of the time data.

Indeed I have not yet found a method for dealing generally with the question of accumulating stress for perfectly indefinite values of t' and τ. At the same time a system of parallel lines on the drawing board enables one readily to find the solution for any definite case. Ten or more loads will be found in a great variety of cases to produce no accumulation, while other cases are easily constructed in which the stress would accumulate enormously were it not for the viscous property of the material. Hence it seems to me that the laws obtained by Wöhler depend upon the rate at which he drove his engine, and that two or three repeated loadings rightly applied might have destroyed the material where several million with a different period completely failed.

11. (*c*) Let us now consider the case where alternate loadings have opposite signs. We have in this case only to change the sign of N and shift the diagram through the interval t'. For generality let us write $N=-N'$. The slightest inspection of the diagrams corresponding to Art. 8 shows that, if the interval of unload be small as compared with that of load, we certainly get accumulation of both positive and negative maxima-stresses, whereas the same time intervals for a repeated load of the *same sense* do not give accumulation. Thus suppose the load N can be safely applied to the bar, that is $2N$ is less than the elastic limit, and let this load be continually repeated without accumulation so that it still remains safe. Now suppose the load N_1 to be alternated with a load $-N_2$. Then the total positive stress, if accumulation takes place, will be $2\,(N_1+N_2)$ which must be less than the elastic limit for safety. This, I think, is the key to Wöhler's law; it is deduced from the ordinary principles of elasticity and involves no recondite characteristic of repeated loading. We see that the 'range of stress' is merely another word in the cases considered for $\frac{1}{2}$ {maximum positive stress}, *i.e* N_1+N_2 and $N+0$; and that the 'accumulation' of the maximum-stresses in the case of alternate directions of loading under the conditions of Wöhler's experiments is the probable explanation of the law propounded by him. The fact that the law can be explained by the ordinary principles of elasticity does not of course detract from the importance we must attribute to its discovery. As an example of the remarks in this paragraph I give Plate II. Fig. IV. (*a*) and (*b*), two diagrams of a load N, followed after a short interval by a load N'. If N and N' are of the same sign (*a*) there is no accumulation, but there is accumulation if they are of opposite sign (*b*).

We note that in the last diagram Fig. IV. (*b*) we not only get the accumulated stresses $\pm 2\,(N+N')$, but also the stresses $\pm(N+2N')$ and $\pm(2N+N')$, which are greater than any which occur with the repeated loads N and N' of the same sign. Similar results involving, however, different conditions for the time intervals may be obtained from diagrams Fig. III. (*b*) and (*c*).

12. (*d*) As tending to throw still more light on Wöhler's experiments I give diagrams for the maximum stresses at the built-in end of the shaft for four alternate loads N and N', and for four alternate loads N and $-N'$ [Fig. V. (*a*) and (*b*) Plate III.]. Underneath these diagrams are those for six alternate loads in the same two cases [Figs. (*c*) and (*d*) Plate III.]. Notwithstanding that these are only special problems, they indicate that very curious results may be obtained by paying regard to the number, duration and sign of successive loads.

Thus we find

(i) $\begin{cases}\text{Two loads } N \text{ and } N'. & \text{Maximum couples } = \pm 2N \text{ or } \pm 2N'. \\ \text{Two loads } N \text{ and } -N'. & \text{Maximum couples } = \pm 2(N+N').\end{cases}$

(ii) Four loads N and N'. Maximum couples $= \pm 2(N+N')$, the same as in two loadings of opposite sign. These maximum couples however do not occur frequently and have only small duration. It is very possible that one or two only will have occurred before viscosity destroys the effect of the first load [Fig. (*a*)].

Four loads N and $-N'$. Maximum couples $= \pm 2(N+N')$ as in the case of only two loadings, *but these couples are of much greater frequency and duration.* In addition *series* of positive and negative couples occur at the built-in end, that is the couples do not so frequently change sign as in the case of four loadings of the same sign [Fig. (*b*)].

(iii) Six loads N and N'. Between the fifth and fourth load there is a small interval in which the maximum couple takes the value $2(N+N')$, but after the sixth load the maximum couples are $\pm 2N$ or $\pm 2N'$. In other words: *a load repeated six times is less likely to destroy the shaft than if it had only been repeated four times* [Fig. (*c*)].

Thus to increase the number of times the load is repeated does not necessarily decrease the safety. It is thus conceivable that four loads judiciously repeated might destroy a shaft, while 43,000,000 applied at stated intervals might leave it still intact. Because a piece of material has withstood millions of repetitions of the same load with a definite period of rotation of the testing machine, it does not follow that it could withstand the same load otherwise repeated and applied.

(iv) Six loads N and $-N'$. Maximum couples $= \pm 2(N+N')$, as in the case of only two or four repeated loads, but the periods over which these couples act have been much extended and the danger to the shaft is correspondingly increased [Fig. (*d*)].

(v) By superposing Figs. (*c*) and (*d*) upon themselves with an interval $= Jt'$, between the first load of the shifted and the last load of the original figure we can obtain diagrams for 12 repeated loads. This in most instances will probably exceed the limit during which the stress-couple due to the first load remains appreciable. In neither case does there appear to be any further tendency to accumulate stress, but rather to diminish the duration of the maximum couples. We may, I think, then fairly conclude that for this particular problem $2(N+N')$ gives the maximum couple in both cases of the same and of alternate sign in loads, but it occurs so rarely and for such short duration in the former case that not improbably $\pm 2N$ or $\pm 2N'$ would be a sufficient limit in practice to take for the maximum couples.

(*e*) The whole of our work and results with regard to repeated loading has been deduced from the solution of equation (ii) in Art. (2). But if E be the stretch modulus of the material of a bar and we put J^2 in that equation equal E/ρ, then, if for ψ we read z the displacement, and for N read T an impulsive load, the whole of our results will apply to a bar suddenly loaded and then unloaded, one end being supposed fixed. Our diagrams thus give results for repeated loading of a bar subject to longitudinal tensile or to alternate longitudinal tensile and compressive loads; we have only to put for N and N' the alternate tensions T and T'.

It will only be necessary to give J a somewhat different value. For iron we may take roughly $E = 27{,}000{,}000$ lbs. per sq. in., and taking a cubic foot as in Art. (5) to weigh 480 lbs. we have:

$$J = 16{,}150 \text{ ft. per sec. about.}$$

Our remarks then on repeated loading apply as much to Wöhler's tensile as to his torsional experiments, for both of which, as for flexure, he states the same general law.

13. Although it has not been possible to deduce a general law regulating repeated loading for all cases we are still, I think, in a position to criticise Wöhler's statements, provided we may assume the phenomena observed by him to be due solely to the already well known properties of elastic material. On p. 6 of his work he writes:

> Das von mir endeckte Gesetz, dessen allgemeine Gültigkeit für Eisen und Stahl durch diese Versuchs-Resultate geprüft werden soll, lautet:
>
> Der Bruch des Materials lässt sich auch durch vielfach wiederholte Schwingungen, von denen keine die absolute Bruchgrenze erreicht herbeiführen.

This is perfectly true, provided the "Bruchgrenze" here denoted is that which would destroy the elasticity of the material by a single sudden loading. A single sudden loading produces of course double the stress due to a gradual statical loading. The rupture owing to vibrations is due to the effect of stress in certain cases of repeated loading tending to accumulate.

> Die Differenzen der Spannungen welche die Schwingungen eingrenzen, sind dabei für die Zerstörung des Zusammenhanges massgebend.
>
> Die absolute Grösse der Grenz-Spannungen ist nur insoweit von Einfluss, als mit wachsender Spannung die Differenzen, welche den Bruch herbeiführen, sich verringern.

I believe these last statements to be only partially correct. The first sentence I think expresses something near the truth in those cases of alternate loadings with opposite sign, in which $\pm 2(N+N')$ or $\pm 2(T+T')$ remain the maximum loads. Here the algebraic difference is *massgebend für die Zerstörung des Zusammenhangs.* But this covers only a certain class of cases in which the alternate loadings have opposite signs. It is easy to arrange a series of loadings in which the stress shall continue to accumulate above $\pm 2(N+N')$ or $\pm 2(T+T')$. It is probable that Wöhler's machine was not, or could not be, run at such a speed as to produce any such series.

Further, I do not think this sentence even if qualified by the second one, expresses the law for alternate loadings of the same sign. In no case that I have come across does $2(N' \sim N)$ or $2(T' \sim T)$ express the maximum load, N and N' being of the same sign. We are thrown back on the case that the repeated loads do not produce more effect than the single loads (*i.e.* no accumulation, maximum-couples $\pm 2N$ or $\pm 2N'$), or else the maximum stress appears to be the same as if the loads had alternate sign {*i.e.* $= 2(N+N')$ or $2(T+T')$} but is of very short duration. Of course it is possible also in this case to so arrange the loadings as to obtain far larger stress-accumulation. The times suitable for such loadings appear to be rare in practice, or we should more frequently hear of a small repeated load destroying a structure. I see no reason, theoretically, why if $2N$ approaches the elastic limit of the material it should be necessary to make $(N-N')$ smaller, as seems indicated in the last sentence of Wöhler's law. Wöhler's result, however, might be obtained if the effect of his machine instead of giving alternate load, N, 0, N', 0, N, etc., practically made the loads continuous, N, N', N, etc. In this case, supposing $N > N'$, we should really be dealing with a continuous statical load N' and a series of loadings $N-N'$, 0, $N-N'$, etc. Without accumulation this would give for the maximum load

$$\begin{aligned} &2(N-N')+N' \\ &= 2N-N'. \end{aligned}$$

And as $2N$ approached the elastic limit it would be necessary to make N' as large as possible, or the difference $N-N'$ as small as possible, thus it might appear that by increasing loading the difference between alternate loads must be diminished*.

---

* There seems a good deal to confirm this, not only in the form of Wöhler's apparatus, but also in the general results of his experiments tabulated on pp. 6 and 7. Wöhler it is true is dealing with rupture and our theory confines itself to the elastic limit. Still in wrought-iron and steel probably any load which fairly exceeds the elastic limit will if repeated often enough "wear out" the material, so that I expect for practical purposes Wöhler's bars were damaged long before the conclusion of the often immense number of repeated loads which preceded actual rupture. Let T_0 be the safe elastic limit of tensile-stress, then for a single load T followed by no second load, we must have $T < T_0/2$; for a single load followed by an equal negative load $-T$, the maximum stress is $2(T+T)$ and we must have $T < T_0/4$; for a load T followed by T', $(T' < T)$ we must have on the above hypothesis as to the nature of Wöhler's machine $(2T-T') < T_0$. Now Wöhler obtains the following numbers as representing safe limits, *i.e.* as giving values not equal to destructive value of stress:

$$\textit{For Iron} \left\{ \begin{array}{lr} T=300, \text{ positive load only;} & \therefore\ T_0 > 600, \\ T=160, \text{ alternate positive and negative loads;} & \therefore\ T_0 > 640, \\ T=440,\ T'=240, \text{ alternate positive loads;} & \therefore\ T_0 > 640. \end{array} \right.$$

Hence allowing for effect of viscosity in diminishing previous maximum stress in case of repeated loads, we see that these give practically the same limit for T_0.

$$\textit{For Cast Steel} \left\{ \begin{array}{lr} T=480, \text{ positive load only;} & \therefore\ T_0 > 960, \\ T=280, \text{ alternate positive and negative loads;} & \therefore\ T_0 > 1120, \\ T=800,\ T'=350, \text{ alternate positive loads;} & \therefore\ T_0 > 1250. \end{array} \right.$$

However this may be, I have little doubt, *that the best practical measure of safety is the ordinary elastic limit, at the same time the 'accumulated stress' must be taken less than this limit. Each problem in repeated load will require to be investigated on its own terms, but so soon as the durations of load and unload are known, the problem can be readily solved.* Such a rule I think ought to replace Wöhler's Law.

14. *Case (v). One terminal of a piece of shafting is fixed, to the other is attached at right angles a lever of length b terminating in a mass m. This mass by means of a blow* ($= mV$), *or otherwise, is started with a velocity V. To investigate the stress in the shafting.*

This case corresponds fairly closely to that of a turn-cock or screw with cross-bar at the top, which we attempt to turn by giving a blow at one end of the cross-bar. We shall treat the cross-bar as rigid and neglect its inertia. There are two sub-cases to be considered. (i) The mass m may be rigidly attached to the terminal of the cross-bar. (ii) The mass is merely placed in contact with it, and leaves it so soon as a tension, not a pressure, is required at the point of contact.

With the previous notation we easily find the following system of equations:

(i) $\dfrac{d^2\psi}{dt^2} = J^2 \dfrac{d^2\psi}{dx^2}$ for all values of x and t for shaft.

(ii) $mb^2 \dfrac{d^2\psi}{dt^2} = +K\dfrac{d\psi}{dx}$ when $x = 0$ for all values of t.

(iii) $\dfrac{d\psi}{dt} = V/b$, when $x = 0$ and $t = \epsilon/J$, the vanishingly small time which the first element of shaft takes to require the spin V/b.

(iv) $\psi = 0$ for all values of x when $t = 0$,

(v) $\psi = 0$ when $x = l$ always.

Now this system of equations is precisely the same as that for the longitudinal impact on a bar, which after attempts by de Saint-Venant and by Sebert and Hugoniot was finally solved in a *finite form* by Boussinesq (de Saint-Venant's

The divergence here is greater but possibly not greater than would be accounted for by the viscosity of axle-steel and the general uncertainty of experiments which *stop short of destroying the bar.*

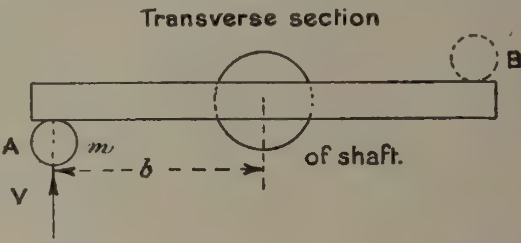
For Spring Cast Steel (Unhardened)	$T = 500$, positive load only;	$\therefore\ T > 1000$.
	Alternate positive loadings.	
	$T = 700,\ T' = 250$	$\therefore\ T_0 > 1150$,
	$T = 800,\ T' = 400$	$\therefore\ T_0 > 1200$,
	$T = 900,\ T' = 600$	$\therefore\ T_0 > 1200$.

This last set seems to confirm my view of Wöhler's apparatus, *i.e.* that it cannot be looked upon as suddenly loading, and again withdrawing load in case of positive repeated loads, *i.e.* $T - T'$ must be looked upon as the variable part of the load.

Clebsch, p. 480*k*). The analytical results of the latter were converted into readily intelligible numerical and graphical forms by de Saint-Venant and Flamant (*Comptes rendus* 1883, pp. 127, 214, 281 and 353). In the following article I make use of their calculations interpreted in the language of torsional stress to mark the characteristics of the above case.

We see that in sub-case (ii), the impact will end when $d\psi/dx=0$. It is to this sub-case (ii) that the curves traced by de Saint-Venant and Flamant can be especially applied and it corresponds to a reversal of the couple at the free end disconnecting it from the mass m.

This seems to me to correspond closely to the nature of a horizontal blow given by a body m in a perpendicular direction to a horizontal bar attached to the top of a vertical bit of shafting. It would also correspond to the case where two heavy masses A and B, each equal to $m/2$, attached to a cross-bar revolving about an axis identical in direction with that of the shaft, strike with spin V/b on opposite sides this cross-head of the shaft. Indeed this case may be treated as covering all impulsive means of putting a shaft in motion *in one direction only.*



Interpreting de Saint-Venant's and Flamant's results in the language of torsional stress we find :—

(1) That the maximum-couple occurs at the built-in end of the shaft.

(2) That it occurs before the end of the impact (*i.e.* before $d\psi/dx=0$, after this the motion becomes periodic).

An examination, however, of the analytical and graphical expressions of the Boussinesq solution shows a series of maxima having one *maximum-maximorum* between the beginning of the impact and the instant when the torsional couple at the free end vanishes. When this torsional couple vanishes we find :

$$Jt=3{\cdot}06767\times l\,;\ =4{\cdot}70856\times l\,;\ =5{\cdot}89974\times l\,;\ =7{\cdot}41874\times l\,;$$

according as $$\gamma=1\,;\ =1/2\,;\ =1/4\,;\ =1/6\,;$$

where $$\gamma=\frac{Kl}{J^2mb^2}=\frac{\rho alk^2}{mb^2}=\frac{\text{second moment of shaft about axis}}{\text{second moment of impulsive load}}.$$

Hence in most practical cases, the head of the impulsive wave will have run three and generally many more times along the shaft, before the torsional couple vanishes. We may thus fairly assume that if there were any maximum after this which *equalled* or *excelled* (*e.g.* with opposite sign) the *maximum-maximorum* before this, viscosity would in all probability have reduced its magnitude to something less or at least to an equality. Thus for both sub-cases we are justified in taking the maximum torsional angles and couples before the vanishing of the torsional-couple as measurements of the chief strains and stresses due to the impulse.

The curves Fig. VIII. (*a*) to (*d*) of Plate IV. are copied and modified from the *Comptes rendus* (1883, *loc. cit.*). They give

(*a*) The torsional angle ψ at the free end from the beginning of the motion up to the first vanishing of the torsional couple at the free end, for the three values of $\gamma=1$, $1/2$, and $1/4$. The ordinate gives $\Psi=(Jb/Vl)\,\psi$.

(*b*) The torsional couple $M=K\,d\psi/dx$ at the built-in end for the same times, and the same values of γ. The ordinate gives $\mathfrak{M}=(Jb/V)\,M/K$.

(*c*) The maximum values ψ_m of ψ at the free end for values of $1/\gamma$ ranging from $1/2$ to 6. The ordinate gives $\Psi_m=(Jb/Vl)\,\psi_m$.

(*d*) The maximum values M_m of $K\,d\psi/dx$ at the built-in end for values of $1/\gamma$ from 0 to 25. The ordinate gives $\mathfrak{M}_m=(Jb/V)\,M_m/K$.

From (*c*) and (*d*) we see that:

For values of $1/\gamma>1$, we may safely take

$$\text{Maximum value of} \qquad \psi=\psi_m=\frac{Vl}{Jb}\sqrt{\frac{1/\gamma}{1+\frac{1}{3}\gamma}},$$

because the value of $\Psi_m=\sqrt{\dfrac{1/\gamma}{1+\frac{1}{3}\gamma}}$ agrees for all practical purposes with its accurately calculated value. The latter being represented by the continuous, the former by the broken curve.

For values of $1/\gamma>5$, we may safely take for *practical* purposes:

$$\text{Maximum couple } M_m=K\frac{V}{Jb}\left\{\sqrt{\frac{1}{\gamma}}+1\right\},$$

because the value of $\mathfrak{M}_m=\sqrt{1/\gamma}+1$ agrees for all practical purposes with its accurately calculated value, the latter being represented by the continuous, the former by the broken curve.

15. *Case* (*vi*). *One terminal of a piece of shafting carries a wheel whose moment of inertia is* $m\beta^2$. *A spin* ω *is given suddenly to this wheel* (*e.g. by an impulsive couple*). *The other end of the shafting is either built-in, or the impulsive stress generated is not sufficient to overcome the frictional, or other resistance at that terminal. To investigate the stress.*

The system of Equations is now

(i) $\dfrac{d^2\psi}{dt^2}=J^2\dfrac{d^2\psi}{dx^2}$ for all values of x and t for shaft.

(ii) $m\beta^2\dfrac{d^2\psi}{dt^2}=+K\dfrac{d\psi}{dx}$ when $x=0$, for all values of t.

(iii) $d\psi/dt=\omega$, when $x=0$ and $t=\epsilon/J$ the vanishingly small time the first element of the shaft takes to acquire the spin ω*.

(iv) $\psi=0$ for all values of x when $t=0$.

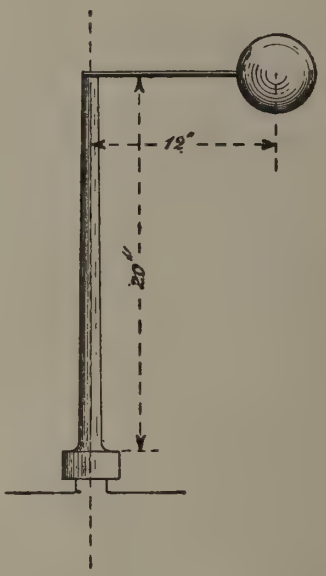
(v) $\psi=0$ when $x=l$ always.

---

* This ω must be distinguished from the same symbol used for area of section of shaft, Art. 3, and now only indirectly involved in $K=\mu\nu$. This area will be denoted by a for the remainder of this paper.

Hence if in the previous case we take $b=\beta$, $V=\beta\omega$, then all the conclusions drawn for sub-case (1) will apply to this new case, for the equational system will be the same for both.

This case obviously includes the practically important problem of the stress produced by suddenly connecting a freely revolving section of shafting with a second section, whose frictional resistance there is not momentum enough to overcome.



16. Examples to Cases (v) and (vi).

Example (*i*). What is the greatest impulsive velocity which can be given to a mass of 10 lbs. connected by a light rigid bar; with a short column or shaft, whose other terminal is fixed, without injuring its elasticity, the dimensions being indicated in the figure annexed?

The bar is of wrought iron and its mean diameter may be taken as one inch.

{3·6 Cubic inches of Wrought Iron weigh 1 lb.}

$$\gamma=\frac{\text{second moment of shaft}}{\text{second moment of ball}}=\frac{\pi\times(\cdot 5)^2\times\dfrac{(\cdot 5)^2}{2}\times 20\times\dfrac{1}{3\cdot 6}}{12^2\times 10}$$

$$=\cdot 00037876=\cdot 00038 \text{ say.}$$

We have next to calculate the safe couple for a shaft of these dimensions S_0 = limit to elastic shearing stress = 20000 lbs. say, $\mu=10{,}500{,}000$, and by equation (v) of Art. 4

$$M_0=\cdot 2821\times 20000\times\pi^{\frac{3}{2}}\times(\cdot 25)^{\frac{3}{2}} \text{ inch lbs.}$$

$$=3927 \text{ inch lbs.}$$

Now the value of γ above is immensely less than 1/5, therefore we may take for the maximum couple

$$M_m=\frac{KV}{bJ}\left(\sqrt{\frac{1}{\gamma}}+1\right).$$

And since we must have $M_m < M_0$, we have for the limiting value of V

$$V_0=J\times\frac{bM_0}{K}\left(\sqrt{\frac{1}{\gamma}}+1\right)^{-1}.$$

Here $$K=10{,}500{,}000\times\pi\times(\cdot 5)^2\times\frac{(\cdot 5)^2}{2}=1030835 \text{ sq. in. lbs.}$$

Therefore $$V_0=10000\times\frac{12\times 3927}{1030835}\div 52\cdot 3825 \text{ ft. per second}$$

$$=8\cdot 73 \text{ ft. per second.}$$

Thus the impulsive momentum (10 lbs. × V) suddenly given to the ball must not exceed 87·3 speed pounds.

The maximum angle through which the ball is displaced

$$
\begin{aligned}
&= \frac{Vl}{Jb}\sqrt{\frac{1/\gamma}{1+\frac{1}{3}\gamma}} \text{ radians} \\
&= \frac{20}{12} \times \frac{8{\cdot}73}{10000} \times 51{\cdot}3828 \text{ radians} \\
&= \frac{{\cdot}01746}{12} \times 51{\cdot}3828 \text{ radians} \\
&= {\cdot}001455 \times 51{\cdot}3828 \times \frac{180}{\pi} \text{ degrees} \\
&= 4^\circ\ 17' \text{ about.}
\end{aligned}
$$

Example (*ii*). A propeller shaft is 30 feet long and of 5 inch diameter. It is of wrought-iron weighing 480 lbs. per cubic foot. Supposing the propeller to be such that its inertia about the axis of the shaft could be represented by a cylindrical belt of metal: inner diameter $=30''$, outer diameter $=36''$, and depth $5''$; weighing 450 lbs. per cubic foot, find the blow which could be given to the propeller at a distance of $18''$ from the axis without injuring the elasticity of the shaft.

The numbers of this problem are purely hypothetical and may differ very considerably from any existing propeller-shaft. The total mass of metal in the cylindrical belt would be about 405 lbs. or ·18 of a ton. I suppose the great majority of propellers would be heavier than this, but it may be doubted whether their inertia would be in proportion to the shaft, much greater than that of the belt above referred to, which amounts to 111,165 sq. in. lbs. In the next place we will suppose the propeller and shafting to be revolving with any given uniform spin, one end of the shafting being maintained at this constant spin throughout the impulse. In this case to calculate the impulsive stress we may suppose the engine end of the shafting to terminate with a fixed cross-section. We should then have to add to the impulsive couple thus deduced the value of any couple in the shaft due to the resistance overcome by the propeller.

An approach, however, to the condition of affairs dealt with in the problem might be reached if we supposed the rotating propeller to be lifted out of the water by the motion of the ship and to receive a blow from the water on entering it again. The uniformity of spin at the engine end of the shaft would only be maintained by the inertia of the engines and is at best a doubtful hypothesis.

I shall first calculate the statical couple required to destroy the elasticity of the shafting in question, we have:

$$
\begin{aligned}
M_0 &= S_0\frac{\pi a^3}{2}, \text{ where } a = \text{radius of shaft-section} \\
&= 20{,}000 \times \pi \times \frac{(2{\cdot}5)^3}{2} \text{ inch lbs.} \\
&= 490{,}874 \text{ inch lbs.}
\end{aligned}
$$

The total angle of torsion corresponding to this

$$=\frac{S_0}{\mu}\times\frac{\text{length of shaft}}{\text{radius}}$$

$$=\frac{2}{1050}\times\frac{12\times 30}{2\cdot 5}\text{ radians}$$

$$=\frac{48}{350\pi}\text{ turns}$$

$$=\cdot 043654\text{ of a turn, } i.e. \text{ about } 1/25 \text{ of a turn}$$

$$=15^\circ\ 43'\text{ about.}$$

This is the angle through which the propeller-end of the shaft could be turned statically without injuring its elasticity.

I shall now calculate the value of $1/\gamma$.

$$1/\gamma=\frac{\pi\left\{\frac{(18)^4}{2}-\frac{(15)^4}{2}\right\}\times 5\times 450}{\pi\times\frac{(2\cdot 5)^4}{2}\times 30\times 12\times 480}$$

$$=\frac{549\times 99\times 5\times 15}{(2\cdot 5)^4\times 30\times 12\times 16}=18\cdot 117.$$

This value of $1/\gamma$ much exceeds 5 and so we may take:

$$M_m=\frac{K\omega}{J}\left\{\sqrt{\frac{1}{\gamma}}+1\right\},$$

$$\psi_m=\frac{l\omega}{J}\left\{\sqrt{\frac{1/\gamma}{1+\frac{1}{3}\gamma}}\right\},$$

ω being the spin impulsively given to the propeller. Putting for M_m the value M_0 and for K, J and $1/\gamma$ their values I find $\omega=2\cdot 77$ revolutions per second, or the impulse must be such as to give not greater speed to the propeller than this.

We find ψ_m by substituting:

$$\psi_m=\frac{30}{10{,}000}\times 2\cdot 77\times\sqrt{\frac{(18\cdot 117)^2}{\frac{1}{3}+18\cdot 117}}\text{ turns}=\cdot 0349\text{ of a turn}$$

$$=12^\circ\ 35'\text{ about.}$$

Thus the angle turned through by the propeller is 20 per cent. less than in the case of statical torsion. Let the blow that will just destroy the elasticity be mv, then $18\times mv=$ moment of momentum $=\omega\times$ moment of inertia of propeller.

$$\therefore\ 18\times mv=\frac{111165}{12}\times 2\pi\times 2\cdot 77,$$

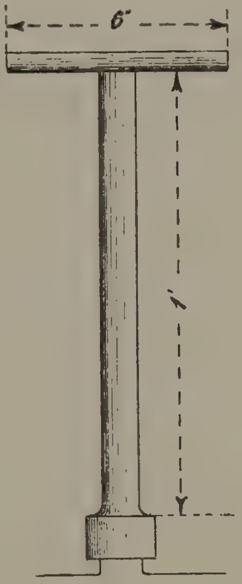
if mv be measured in foot-second-pound scale.

We easily deduce, $mv=8957\cdot 2$ speed pounds.

Now a cubic foot of water weighs $\frac{1000}{16}$ lbs. $=62{\cdot}5$ lbs., or 143 cubic feet of water falling with a speed of 1 foot per second on a point of the propeller distant 1·5 feet from the axis*, would give a blow sufficient to injure the elasticity of the shaft, —always supposing the propeller to absorb the complete momentum of the cubic foot of water.

If the numbers adopted for this problem, as well as its hypothesis of practical uniformity of spin at the engine end be even approximately true, the danger to the elasticity of propeller shafting owing to the falling of masses of water upon the propeller seems very considerable.

17. *Example* (*iii*). A key the head of which is 6″ and the shaft 1 foot is made of wrought iron, with a uniform cross section of diameter 1″ in shaft and ·5″ in the head.



Supposing an impulsive couple whose moment of momentum $=1$ in foot-lbs.-sec. units to be applied to the head of the key and fail to turn it, would this injure the elasticity of the shaft?—We suppose the flexure of the head may be neglected.

Let us first find $1/\gamma$. The swing radius of head $=\left(\frac{3^2}{3}\right)^{\frac{1}{2}}=\sqrt{3}$ nearly.

$$1/\gamma=\frac{\pi(1/4)^2\times 6\times 3}{\pi({\cdot}5)^2\times 12\times\frac{({\cdot}5)^2}{2}}=3.$$

Hence referring to our Fig. VIII. (d), we see that $M_m=2{\cdot}924$. Thus the maximum couple is given by

$$M_m=\frac{K\omega}{J}\times 2{\cdot}924.$$

Here $K=\mu\times\pi\frac{a^4}{2}$ where a is the radius of the shaft.

Now the maximum statical couple is

$$M_0=S_0\frac{\pi a^3}{2},$$

and thus for the limiting spin ω_0 we have:

$$S_0\frac{\pi a^3}{2}=\mu\times\frac{\pi a^4}{2}\times\frac{\omega_0}{J}\times 2{\cdot}924,$$

$$\therefore\ \omega_0=\frac{S_0}{\mu}\times\frac{J}{a}\times\frac{1}{2{\cdot}924}\text{ radians per sec.}$$

$$=\frac{2}{1050}\times\frac{10000\times 12}{{\cdot}5}\times\frac{1}{2{\cdot}924}=156{\cdot}3\text{ radians per sec.}$$

---

* Or roughly a cube of water of 2 feet side, falling about 5 feet distance upon the propeller would destroy the elasticity of the shaft. Such an impact does not seem improbable in a rough sea.

Now wrought iron weighs about 1 lb. to the 3·6 cubic inches. Hence the moment of inertia of the head of the key

$$=\frac{\frac{9}{3}\times\pi\times(\frac{1}{4})^2\times 6}{3\cdot 6}=\frac{\pi}{3\cdot 2}\text{ in sq. inch lbs.}$$

Thus we have by the principle of angular momentum

$$\frac{\pi}{3\cdot 2}\Omega=1\times 12\times 12,$$

where Ω is the spin produced in the key-head by the impulsive couple. It follows that

$$\Omega=\frac{144\times 3\cdot 2}{\pi}\text{ radians, or }\Omega\text{ is}<\omega_0.$$

Thus the couple will not destroy the elasticity of the shaft of the key, but one very slightly greater (namely, of about 1·066 units) would do so. We see that comparatively small impulsive forces are needed to destroy the elasticity of such a shaft.

18. *Case (vii). One end of a length l of shafting is maintained at a constant spin ω [(e.g. by being connected with a massive wheel rotating with this spin): at the other end of this shafting is a wheel of moment of inertia $m\beta^2$ which together with the shafting is initially rotating with the same spin ω. Suddenly a frictional couple N is applied to the wheel: to determine the impulsive stress in the shafting.*

Measuring x from the end where the friction is to be overcome, we have the following system of equations:

$$\frac{d^2\psi}{dt^2}=J\frac{d^2\psi}{dx^2}\quad\text{......(i)},$$

$$m\beta^2\frac{d^2\psi}{dt^2}=K\frac{d\psi}{dx}-N,\text{ when }x=0\text{ for all values of }t>0\quad\text{......(ii)},$$

$$\frac{d\psi}{dt}=\omega,\text{ when }x=l,\text{ for all values of }t\quad\text{......(iii)},$$

$$\frac{d\psi}{dx}=0,\text{ when }t<0,\text{ for all values of }x\text{ from }0\text{ to }l\quad\text{......(iv)},$$

$$\frac{d\psi}{dt}=\omega\quad\text{......(v)}.$$

As a first method of solution, let us employ that of arbitrary functions, and take

$$\psi=\omega t+f(Jt-x)-f(Jt+x-2l)\quad\text{......(vi)},$$

which satisfies (i) and (iii). We have

$$\left.\begin{aligned}\frac{d\psi}{dt}&=\omega+J\{f'(Jt-x)-f'(Jt+x-2l)\}\\ \frac{d\psi}{dx}&=-f'(Jt-x)-f'(Jt+x-2l)\end{aligned}\right\}\text{..........................(vii)},$$

and initially or for $t<0$

whence we may take $$\left.\begin{aligned}f'(\zeta)&=0 \text{ for } \zeta<0\\ f(\zeta)&=0 \text{ for } \zeta=0\end{aligned}\right\}\text{..............................(viii)}.$$

We have now to satisfy (ii) by substituting (vi) in it. Putting $Jt=\zeta$ and $K/m\beta^2J^2=p$, we have

$$f''(\zeta)-f''(\zeta-2l)=p\{-f'(\zeta)-f'(\zeta-2l)\}-\frac{N}{K}p,$$

$$f''(\zeta)+pf'(\zeta)=f''(\zeta-2l)-pf'(\zeta-2l)-\frac{N}{K}p.$$

Integrating,

$$f'(\zeta)+pf(\zeta)=f'(\zeta-2l)-pf(\zeta-2l)-\frac{Np\zeta}{K}+C.$$

But when $\zeta=Jt=0$, $f'(\zeta)$, $f(\zeta)$, $f'(\zeta-2l)$, and $f(\zeta-2l)$ all vanish, hence $C=0$.

Multiply by $\epsilon^{p\zeta}$ and integrate:

$$f(\zeta)\,\epsilon^{p\zeta}=\int\epsilon^{p\zeta}\{f'(\zeta-2l)-pf(\zeta-2l)\}\,d\zeta-\frac{N}{Kp}\,\epsilon^{p\zeta}(p\zeta-1)+C'.$$

Take the limits between 0 and ζ and we have

$$f(\zeta)=\epsilon^{-p\zeta}\int_0^\zeta\epsilon^{p\zeta}\{f'(\zeta-2l)-pf(\zeta-2l)\}\,d\zeta-\frac{N}{Kp}(p\zeta-1+\epsilon^{-p\zeta}).$$

But $f(\zeta-2l)$ and $f'(\zeta-2l)=0$, if $\zeta<2l$, therefore the lower limit of the integral can be changed to $2l$. Put $\eta=p\zeta$, $\gamma=pl=$ ratio of inertia of shaft to that of wheel, and $\sigma=\eta-2\gamma$, and we find:

$$\begin{aligned}f(\zeta)&=\frac{Nl}{K\gamma}(1-\eta-\epsilon^{-\eta})+f(\zeta-2l)-2\epsilon^{-\eta}\int_{2\gamma}^{\eta}\epsilon^{\eta}f(\eta-2\gamma)\,d\eta\\ &=\frac{Nl}{K\gamma}(1-\eta-\epsilon^{-\eta})+f(\zeta-2l)-2\epsilon^{-\sigma}\int_0^{\sigma}\epsilon^{\sigma}f(\sigma)\,d\sigma,\end{aligned}$$

or finally,

$$f(\zeta)=\frac{Nl}{K\gamma}(1-\eta-\epsilon^{-\eta})+f(\sigma)-2\epsilon^{-\sigma}\int_0^{\sigma}\epsilon^{\sigma}f(\sigma)\,d\sigma\quad\text{......................(ix)},$$

where $\sigma=\eta-2\gamma$, $f(\sigma)=f(\zeta-2l)$, and we have thus the means of determining any run of values $2l$ of $f(\zeta)$ from the previous run of $2l$.

19. I have obtained the following series of values of $f(\zeta)$ by calculation from equation (ix).

$$(a)\quad f(\zeta<0)=0,$$
$$f'(\zeta<0)=0.$$

$$(b)\quad f\left(\zeta=\genfrac{}{}{0pt}{}{2l}{0}\right)=\frac{Nl}{K\gamma}(1-\eta-\epsilon^{-\eta}),$$
$$f'\left(\zeta=\genfrac{}{}{0pt}{}{2l}{0}\right)=\frac{N}{K}(\epsilon^{-\eta}-1).$$

$$(c)\quad f\left(\zeta=\genfrac{}{}{0pt}{}{4l}{2l}\right)=\frac{Nl}{K\gamma}(1-\eta-\epsilon^{-\eta})+\frac{Nl}{K\gamma}\{(\eta-2\gamma)-3+3\epsilon^{-(\eta-2\gamma)}+2(\eta-2\gamma)\,\epsilon^{-(\eta-2\gamma)}\},$$
$$f'\left(\zeta=\genfrac{}{}{0pt}{}{4l}{2l}\right)=\frac{N}{K}(\epsilon^{-\eta}-1)+\frac{N}{K}\{1-\epsilon^{-(\eta-2\gamma)}-2(\eta-2\gamma)\,\epsilon^{-(\eta-2\gamma)}\}.$$

$$(d)\quad f\left(\zeta=\genfrac{}{}{0pt}{}{6l}{4l}\right)=f\left(\zeta=\genfrac{}{}{0pt}{}{4l}{2l}\right)+\frac{Nl}{K\gamma}\{5-(\eta-4\gamma)-[5+4(\eta-4\gamma)$$
$$+2(\eta-4\gamma)^2]\times\epsilon^{-(\eta-4\gamma)}\},$$
$$f'\left(\zeta=\genfrac{}{}{0pt}{}{6l}{4l}\right)=f'\left(\zeta=\genfrac{}{}{0pt}{}{4l}{2l}\right)+\frac{N}{K}\{-1+[1+2(\eta-4\gamma)^2]\,\epsilon^{-(\eta-4\gamma)}\}.$$

$$(e)\quad f\left(\zeta=\genfrac{}{}{0pt}{}{8l}{6l}\right)=f\left(\zeta=\genfrac{}{}{0pt}{}{6l}{4l}\right)+\frac{Nl}{K\gamma}\{(\eta-6\gamma)-7+[7+6(\eta-6\gamma)+2(\eta-6\gamma)^2$$
$$+\tfrac{4}{3}(\eta-6\gamma)^3]^{-(\eta-6\gamma)}\},$$
$$f'\left(\zeta=\genfrac{}{}{0pt}{}{8l}{6l}\right)=f'\left(\zeta=\genfrac{}{}{0pt}{}{6l}{4l}\right)+\frac{N}{K}\{1-[1+2(\eta-6\gamma)-2(\eta-6\gamma)^2$$
$$+\tfrac{4}{3}(\eta-6\gamma)^3]\,\epsilon^{-(\eta-6\gamma)}\},$$
$$f\left(\zeta=\genfrac{}{}{0pt}{}{10l}{8l}\right)=f\left(\zeta=\genfrac{}{}{0pt}{}{8l}{6l}\right)+\frac{Nl}{K\gamma}\{9-(\eta-8\gamma)-[9+8(\eta-8\gamma)+4(\eta-8\gamma)^2$$
$$+\tfrac{2}{3}(\eta-8\gamma)^4]\,\epsilon^{-(\eta-8\gamma)}\},$$
$$f'\left(\zeta=\genfrac{}{}{0pt}{}{10l}{8l}\right)=f'\left(\zeta=\genfrac{}{}{0pt}{}{8l}{6l}\right)+\frac{N}{K}\{-1+[1+4(\eta-8\gamma)^2-\tfrac{8}{3}(\eta-8\gamma)^3$$
$$+\tfrac{2}{3}(\eta-8\gamma)^4]\,\epsilon^{-(\eta-8\gamma)}\}.$$

20. It is easy to carry the analytical expressions for $f(\zeta)$ and $f'(\zeta)$ further, but I have not done so on account of the laborious nature of the calculations for special numerical values of γ, without which the analytical expressions are practically useless. In Table I (a) at the end of this memoir I give the values of $\phi(\zeta)=(K\gamma/Nl)f(\zeta)$ and $\phi'(\zeta)=(K/N)f'(\zeta)$ from $\zeta=0$ to $10l$ for the values 1, 1/2, 1/10, 1/100 and 1/1000 of γ. In the construction of this table I have been assisted by Newman and Glaisher's *Tables of the Descending Exponential* (*Camb. Phil. Trans.* Vol. XIII., pp. 145—272), but many exponentials sought had to be deduced by additional calculation, as the entries in their tables only go to the second place of decimals in the argument. The labour of determining the values of $\phi(\zeta)$ and $\phi'(\zeta)$ much exceeded my anticipations and probably was hardly justified by the practical value of the results. At the same time it is interesting to note how the curves of spin and stress calculated by means of the exponentials give more abrupt changes than those obtained from the superposition of the two or three principal harmonics: see Art. 40. The values of the function $f(\zeta)$ and its slope $f'(\zeta)$ have been calculated and revised with much care, and I believe the plotted diagrams will be found sensibly correct for all practical purposes. The limitations to the range of values for ζ only allows in the case of a large value of γ more than one maximum to be observed, which is not necessarily the *maximum-maximorum*. Allowing for the effect of viscosity, however, it is highly probable that the *maxima* which occur after $Jt = 10l$ do not sensibly exceed that which precedes this instant. At the same time definite experiments on this point would be of very considerable value. I shall indicate in the sequel of this paper, however, how a superior limit to the *maximum-maximorum* on the hypothesis of no viscosity can be readily obtained.

21. The graphical results given on Plate V. for $\gamma = 1/100$ and $\gamma = 1/1000$ do not differ by visible amounts on the paper from what we obtain by the method of a later section of this memoir (p. 57), *i.e.* by considering the shaft under statical torsion at each instant. Thus

$$2\phi'(\zeta) = 1 - \cos\sqrt{\gamma}\,(\zeta/l + 1)$$

to three and five places of decimals respectively when we compare the tabulated values of $\phi'(\zeta)$ in Table I for $\gamma = 1/100$ and $1/1000$ with those calculated for the right-hand side of the above relation. The investigation thus shews that for practical purposes we may treat shafts for which $\gamma = 1/100$ and upwards, and these are the most frequent, by the method of Art. 41. The curves of torsion and stress then become such as we have plotted on Plate XI. Fig. XVI. Case (β), or reduce sensibly to simple trigonometrical terms. The maximum stress-couple is then $= 2N$, and the only noteworthy point is the long interval which elapses before that maximum is reached, *i.e.* it is not reached until

$$Jt/l = \pi/\sqrt{\gamma} = 41{\cdot}416 \text{ for } \gamma = 1/100, = 99{\cdot}346 \text{ for } \gamma = 1/1000.$$

Thus, in the latter case the wave must run nearly 100 times along the shaft, and during this interval viscosity will generally have had a very sensible influence in reducing the amplitude of the wave and so the maximum stress.

The curves for $\phi(\zeta)$ and $\phi'(\zeta)$ are interesting as they exhibit the gradual change of form of those functions as they sensibly approach to a simple trigonometrical term, when $\gamma = 1/1000$ and less.

In the following diagrams and tables I shall only deal with the more theoretically interesting if less frequent practical cases of $\gamma = 1$, $1/2$ and $1/10$.

22. The diagrams as they stand exhibit some interesting results, of which I note the following:

(i) At the pulley-end of the shaft, the angle of torsion never changes its sign, although it has a greater range of values than at any point nearer the statical end (Plate VI.).

(ii) The angle of torsion has a greater value for $\gamma=1$ than for $\gamma=1/2$ or $1/10$ at the $1/4$, $1/2$ and $3/4$ length of shaft, but a less value than $\gamma=1/2$ at the pulley end. Thus to increase the inertia of the shaft relative to that of the pulley only lessens the angle of torsion at the pulley-end (Plate VII.).

(iii) The variations in spin at the pulley-end,—an important point in the case of certain weaving machinery—increase as we increase the inertia of the shaft. The variation in spin at all points of the shaft is alternately negative and positive, and is greatest at the pulley-end of the shaft.

At the pulley-end of the shaft we have for spin variations:

$$\gamma=1, \qquad -\cdot865\times\frac{30JN}{\pi K} \text{ and } +\cdot85\times\frac{30JN}{\pi K} \text{ revolutions per min.}$$

$$\gamma=1/2, \qquad -\cdot65\times\frac{30JN}{\pi K} \text{ and } +\cdot65\times\frac{30JN}{\pi K} \quad ,, \quad ,, \quad ,,$$

$$\gamma=1/10 \qquad -\cdot36\times\frac{30JN}{\pi K} \text{ and } +\cdot36\times\frac{30JN}{\pi K} \quad ,, \quad ,, \quad ,,$$

(iv) The stress-couple has its maximum at the *rotated end* of the shaft, in other words the shaft is more likely to be destroyed at the end at which work is put into it, than at the end at which work is taken out,—at the engine end rather than at the end thrown into gear.

At the same time the difference in the stress-couples at the ends of a shaft becomes small when the inertia of the pulley is greater as compared with that of the shaft: it is still sensible for $\gamma=1/10$, but hardly for $\gamma=1/100$. For $\gamma=1$, however, it amounts to as much almost as 2/5 of the applied couple.

(v) At the pulley-end the stress-couple cannot be negative. It has a very slight negative value at a quarter length of span from this end, and from this point to the rotated end of the shaft very sensible negative values for large values of γ, but tends to have only sensible positive values for $\gamma=1/10$ and less.

(vi) The dynamical stress-couple largely exceeds the statical couple applied to the pulley, and in the case of γ being large is considerably greater than its double, a value which might possibly have been assumed for it on the basis of similar theorems in kinetic stress due to Young, Tredgold, and Poncelet.

In conclusion I may note that if Wöhler's views on the influence of *range* as distinguished from magnitude of stress were correct, then the comparative weakness of a shaft under kinetic stress would be still further magnified, owing to the stress-couple taking negative values.

23. *Case (viii). One end of a length l of shafting is suddenly started with and maintained at a spin ω; at the other end of the shafting is a wheel of moment of inertia $m\beta^2$, which together with the shafting is initially at rest, but when revolving it has to overcome a frictional couple N'; to determine the impulsive couple in the shafting.*

I shall divide this problem into two parts: (i) to consider what happens before the impulsive wave produced by the spin ω given to one end of the shaft reaches the wheel at the other, (ii) to discuss the motion after the impulse reaches the wheel.

This division of the problem is necessary because the couple N' does not commence to act until the impulse has reached the wheel, *i.e.* at the time $t = l/J$.

(i) Since to start with there is only a wave in one direction we choose

$$\psi = f(Jt - x')$$

where x' is measured from the end revolving with spin ω.

Initially $\dfrac{d\psi}{dt} = 0$ for all values of x' from 0 to l,

$= \omega$ for all negative values of x',

$\therefore\ Jf'(-x') = 0$ unless x' is negative or zero, and then it equals ω;

$\therefore\ f(\zeta) = 0$, unless ζ is positive, and then it equals $\dfrac{\omega}{J}\zeta$.

Thus we see that $\psi = 0$ unless $Jt > x'$ and then it equals $\dfrac{\omega}{J}(Jt - x')$.

Hence when $Jt = l$ we have

$$\frac{d\psi}{dt} = \omega,\ \frac{d\psi}{dx} = -\frac{\omega}{J} \text{ for all values of } x' \text{ from 0 to } l.$$

This leads us to an important consideration with regard to shafting, namely, if one end of it be suddenly started with a spin ω, then for safety, it is necessary that it should be capable of withstanding a greater maximum couple than $M = K\omega/J$.

This result is quite independent of the length of the shafting and arises solely from the impulse given to the shaft by starting one end with the impulsive spin ω.

The importance of a condition of this kind for the safety of the shafting of machinery suddenly thrown into gear can hardly be over estimated.

24. As an example let us calculate the maximum spin which can be *suddenly* given to the end of a shaft 2·5″ diameter without injuring its elasticity; the material of the shaft being wrought iron.

By Equation (v) of Art. 4, the maximum value of the torsional couple for a circular sectional shaft is $M_0 = {\cdot}2821\, S_0 \sqrt{\alpha^3}$, α being the area.

Further, supposing we may assume the safe tensile load for wrought iron to be $T_0 = 25000$ lbs. per sq. in. we should have on the uniconstant hypothesis:

$$S_0 = \frac{4}{5}\, T_0 = 20000 \text{ lbs. per sq. in.}$$

Next $K = \mu\nu = \mu \times \alpha \times \tfrac{1}{2}\left(\frac{2{\cdot}5}{2}\right)^2$ and taking $\mu = 10{,}500{,}000$ lbs. per sq. in. as before, we have

$$10{,}500{,}000 \times \alpha \times \frac{(2{\cdot}5)^2}{8} \times \frac{\omega}{J} = {\cdot}2821 \times 20000 \times \alpha\sqrt{\pi} \cdot \frac{2{\cdot}5}{2}\, .$$

Hence $$\omega = \frac{10000 \times 12 \times 4 \times {\cdot}2821 \times 20\sqrt{\pi}}{2{\cdot}5 \times 10500} \times \frac{1}{2\pi} \text{ revolutions per sec.}$$

since $J = 10000$ ft. per sec. Therefore $\omega = 29{\cdot}12$ revolutions per sec. about. I shall investigate in *Case* (x), the effect on this result of *gradually* getting up speed, we shall see that it is very sensibly modified.

25. (ii) The result of (i) enables us to measure our time from a new epoch and to start with initial conditions,

$$\frac{d\psi}{dt} = \omega,\quad \frac{d\psi}{dx} = \frac{\omega}{J},$$

(measuring x from the wheel end of the shaft now), at the instant the couple N' is first applied. Our system of equations now becomes:

$$\frac{d^2\psi}{dt^2} = J^2 \frac{d^2\psi}{dx^2} \text{...(i)},$$

$$m\beta^2 \frac{d^2\psi}{dt^2} = K\frac{d\psi}{dx} - N', \text{ when } x = 0 \text{ for all values of } t > 0 \text{ ....(ii)},$$

$$\frac{d\psi}{dt} = \omega, \text{ when } x = l \text{ for all values of } t \text{..........................(iii)},$$

$$\frac{d\psi}{dt} = \omega, \text{ when } t < 0 \text{ for all values of } x \text{ from } 0 \text{ to } l \text{ ....(iv)},$$

$$\frac{d\psi}{dx} = \omega/J \text{ ....(v)},$$

(i) and (iii) give us as solution

$$\psi = \omega\left(t + \frac{x - l}{J}\right) + f(Jt - x) - f(Jt + x - 2l) \text{.....................(vi)},$$

with the conditions from (iv) and (v) that

$$f(\zeta < 0) = 0 \text{ and } f'(\zeta < 0) = 0 \text{ ....(vii)}.$$

Substitute (vi) in (ii) and we find:

$$m^2\beta^2J^2\{f''(\zeta)-f''(\zeta-2l)\}=K\left\{-f'(\zeta)-f'(\zeta-2l)+\frac{\omega}{J}\right\}-N'$$

or,

$$f''(\zeta)+pf'(\zeta)=f''(\zeta-2l)-pf'(\zeta-2l)+p\left(\frac{\omega}{J}-\frac{N'}{K}\right).$$

Integrating and remembering (vii) we find

$$f'(\zeta)+pf(\zeta)=f'(\zeta-2l)-pf(\zeta-2l)-\left(\frac{N}{K}-\frac{\omega}{J}\right)p\zeta.$$

This agrees exactly with the equation of the previous case (p. 27) if we write $N'-K\omega/J$ for N. With this modification therefore the series of values for $f(\zeta)$, (a) to (e) apply to this case, and the corresponding values of the stress-couple will be obtained from the previous results, if we put $N'-K\omega/J$ for N in the values obtained and add $K\omega/J$ to them.

26. *Case (ix). An axle of length 2l terminating in a pair of wheels, each of moment of inertia $m\beta^2$, forms a system rotating with spin ω. Suddenly a retarding couple N is applied to each wheel: it is required to determine the stress in the axle.*

If the shearing stress due to a superincumbent load be added to the torsional stress at each terminal of the axle we have a very fair theoretical representation of the stress in a railway carriage axle, when the brakes are being applied to the wheels, but a still more important application is to plate-rolling mills.

The system of equations now takes the following form:

$$\frac{d^2\psi}{dt^2}=J^2\frac{d^2\psi}{dx^2}\text{ .... (i)},$$

$$m\beta^2\frac{d^2\psi}{dt^2}=K\frac{d\psi}{dx}-N\text{ at both terminals .... (ii)},$$

$$\frac{d\psi}{dt}=\omega\text{ for all values of } x\text{, if } t \text{ be less than } 0\text{ .... (iii)},$$

$$\frac{d\psi}{dx}=0\text{ .... (iv)}.$$

Now it is obvious that everything must be symmetrical about the centre of the axle, or if we were to take that point as origin of x, the solution of (i) must be of the form

$$\psi=\omega t+f(Jt-x)+f(Jt+x).$$

Hence, if we take one terminal as origin, we have

$$\psi=\omega t+f(Jt+x-l)+f(Jt+l-x)\text{ .... (v)},$$

as a proper form of solution.

Applying the conditions (iii) and (iv) to (v) we find

$$f'(x-l) = f'(l-x) = 0,$$

hence we deduce as before

$$f\left(\zeta = \genfrac{}{}{0pt}{}{+l}{-l}\right) = 0, \qquad f'\left(\zeta = \genfrac{}{}{0pt}{}{+l}{-l}\right) = 0 \text{ ....(vi).}$$

Substituting (v) in (ii) we have for $x = 0$

$$f''(Jt+l) + f''(Jt-l) = p\{f'(Jt-l) - f'(Jt+l)\} - \frac{Np}{K},$$

or integrating and putting $Jt = \zeta$,

$$f'(\zeta+l) + pf(\zeta+l) = pf(\zeta-l) - f'(\zeta-l) - \frac{Np\zeta}{K},$$

the constant vanishing owing to the relations (vi) when $\zeta = 0$.

Integrating a second time we obtain, if $\eta = p\zeta$ as before,

$$f(\zeta+l) = \frac{Nl}{K\gamma}(1-\eta-\epsilon^{-\eta}) - f(\zeta-l) + 2\epsilon^{-\eta}\int_{2\gamma}^{\eta} \epsilon^{\eta} f(\eta-\gamma)\, d\eta \text{ ....(vii).}$$

Since $f(\eta-\gamma) = 0$ for all values from 0 to 2γ, so that the lower limit of the integral may be taken as 2γ instead of 0, (vii) gives us the law of succession of the various $2l$ ranges of the function $f(\zeta)$ and we note that they closely resemble those of case (vii). I tabulate them below.

27. (a) $f\left(\zeta = \genfrac{}{}{0pt}{}{+l}{-l}\right) = 0,$

$f'\left(\zeta = \genfrac{}{}{0pt}{}{+l}{-l}\right) = 0.$

(b) $f\left(\zeta = \genfrac{}{}{0pt}{}{3l}{l}\right) = \dfrac{Nl}{K\gamma}\{1-(\eta-\gamma)-\epsilon^{-(\eta-\gamma)}\},$

$f'\left(\zeta = \genfrac{}{}{0pt}{}{3l}{l}\right) = \dfrac{N}{K}\{\epsilon^{-(\eta-\gamma)}-1\}.$

(c) $f\left(\zeta = \genfrac{}{}{0pt}{}{5l}{3l}\right) = f\left(\zeta = \genfrac{}{}{0pt}{}{3l}{l}\right) - \dfrac{Nl}{K\gamma}\{(\eta-3\gamma)-3+[3+2(\eta-3\gamma)]\,\epsilon^{-(\eta-3\gamma)}\},$

$f'\left(\zeta = \genfrac{}{}{0pt}{}{5l}{3l}\right) = f'\left(\zeta = \genfrac{}{}{0pt}{}{3l}{l}\right) - \dfrac{N}{K}\{1-[1+2(\eta-3\gamma)]\,\epsilon^{-(\eta-3\gamma)}\}.$

(d) $f\left(\zeta = \genfrac{}{}{0pt}{}{7l}{5l}\right) = f\left(\zeta = \genfrac{}{}{0pt}{}{5l}{3l}\right) + \dfrac{Nl}{K\gamma}\{5-(\eta-5\gamma)-[5+4(\eta-5\gamma)$

$+2(\eta-5\gamma)^2]\,\epsilon^{-(\eta-5\gamma)}\},$

$f'\left(\zeta = \genfrac{}{}{0pt}{}{7l}{5l}\right) = f'\left(\zeta = \genfrac{}{}{0pt}{}{5l}{3l}\right) + \dfrac{N}{K}\{-1+[1+2(\eta-5\gamma)^2]\,\epsilon^{-(\eta-5\gamma)}\}.$

$$(e)\quad f\left(\zeta = \begin{matrix}9l\\7l\end{matrix}\right) = f\left(\zeta = \begin{matrix}7l\\5l\end{matrix}\right) - \frac{Nl}{K\gamma}\{(\eta - 7\gamma) - 7 + [7 + 6(\eta - 7\gamma) + 2(\eta - 7\gamma)^2 + \tfrac{4}{3}(\eta - 7\gamma)^3]\,\epsilon^{-(\eta - 7\gamma)}\},$$

$$f'\left(\zeta = \begin{matrix}9l\\7l\end{matrix}\right) = f'\left(\zeta = \begin{matrix}7l\\5l\end{matrix}\right) - \frac{N}{K}\{1 - [1 + 2(\eta - 7\gamma) - 2(\eta - 7\gamma)^2 + \tfrac{4}{3}(\eta - 7\gamma)^3]\,\epsilon^{-(\eta - 7\gamma)}\}.$$

$$(f)\quad f\left(\zeta = \begin{matrix}11l\\9l\end{matrix}\right) = f\left(\zeta = \begin{matrix}9l\\7l\end{matrix}\right) + \frac{Nl}{K\gamma}\{9 - (\eta - 9\gamma) - [9 - 8(\eta - 9\gamma) + 4(\eta - 9\gamma)^2 + \tfrac{2}{3}(\eta - 9\gamma)^4]\,\epsilon^{-(\eta - 9\gamma)}\},$$

$$f'\left(\zeta = \begin{matrix}11l\\9l\end{matrix}\right) = f'\left(\zeta = \begin{matrix}9l\\7l\end{matrix}\right) + \frac{N}{K}\{-1 + [1 + 4(\eta - 9\gamma)^2 - \tfrac{8}{3}(\eta - 9\gamma)^3 + \tfrac{2}{3}(\eta - 9\gamma)^4]\,\epsilon^{-(\eta - 9\gamma)}\}.$$

$$(g)\quad f\left(\zeta = \begin{matrix}13l\\11l\end{matrix}\right) = f\left(\zeta = \begin{matrix}11l\\9l\end{matrix}\right) - \frac{Nl}{K\gamma}\{(\eta - 11\gamma) - 11 + [11 + 10(\eta - 11\gamma) + 4(\eta - 11\gamma)^2 + \tfrac{8}{3}(\eta - 11\gamma)^3 - \tfrac{2}{3}(\eta - 11\gamma)^4 + \tfrac{4}{15}(\eta - 11\gamma)^5]\,\epsilon^{-(\eta - 11\gamma)}\},$$

$$f'\left(\zeta = \begin{matrix}13l\\11l\end{matrix}\right) = f'\left(\zeta = \begin{matrix}11l\\9l\end{matrix}\right) - \frac{N}{K}\{1 - [1 + 2(\eta - 11\gamma) - 4(\eta - 11\gamma)^2 + \tfrac{16}{3}(\eta - 11\gamma)^3 - \tfrac{6}{3}(\eta - 11\gamma)^4 + \tfrac{4}{15}(\eta - 11\gamma)^5]\,\epsilon^{-(\eta - 11\gamma)}\}.$$

I have carried the formulae one step further than in *Case* (*vii*), but they do not represent even thus as wide a range of values, as the axle is $2l$, and the shaft l feet long.

28. Table I (b) at the end of this memoir gives the values of

$$\phi(\zeta) = \frac{K\gamma}{Nl} f(\zeta) \text{ and } \phi'(\zeta) = \frac{K}{N} f'(\zeta),$$

for $\gamma = 1$, $1/2$, $1/10$, $1/100$ and $1/1000$, for values of the variable $\zeta = -l$ to $11l$. In Plate VIII. Fig. XII. (a) are plotted the values of $\phi(\zeta)$ for $\gamma = 1, \frac{1}{2}, \frac{1}{10}, \frac{1}{100}$, and in Fig. XII. ($b$) of $\phi'(\zeta)$, for $\gamma = 1, \frac{1}{2}, \frac{1}{10}$; the others are too close to the x-axis to be represented on our scale. Indeed for axles whose inertia relative to the wheels is so small, the stress produced by sudden 'braking' is also very small, *i.e.* the torsional couple is a very small fraction of the brake-couple.

Before considering the results obtained, I will first write down the solution of the problem on the hypothesis that the axle and wheels form a rigid body. We have for the equation of motion

$$(m\beta^2 + \gamma m\beta^2)\frac{d^2\psi}{dt^2} = -N.$$

Hence

$$\dot{\psi} - \omega = -\frac{N}{m\beta^2(1+\gamma)}t,$$

$$\psi - \omega t = -\frac{N}{m\beta^2(1+\gamma)} \cdot \frac{t^2}{2}.$$

But
$$J^2 = \frac{Kl}{\gamma m \beta^2},$$

$$\therefore\ \psi - \omega = -\frac{NJ}{K} \cdot \frac{\gamma}{1+\gamma} \cdot \frac{Jt}{l},$$

$$\psi - \omega t = -\frac{Nl}{K} \cdot \frac{\gamma}{2(1+\gamma)} (Jt/l)^2.$$

Thus the spin is a linear function, and the angle of rotation a parabolic function of the time. We should expect therefore as γ gets small that the values found for ψ on the supposition of the elastic nature of the axle would approach these functions. In our solution we have for the elastic axle

$$\dot{\psi} - \omega = \frac{NJ}{K} \{\phi'(Jt - x) + \phi'(Jt + x)\},$$

$$\psi - \omega t = \frac{Nl}{K} \left\{\frac{\phi(Jt - x)}{\gamma} + \frac{\phi(Jt + x)}{\gamma}\right\}.$$

For $x = 0$, (or for Jt very great compared with x supposing ϕ and ϕ' to be non-periodic,) the terms within the brackets become

$$2\phi'(Jt) \text{ and } \frac{2\phi(Jt)}{\gamma}.$$

Hence we should expect for large values of γ, that

$$\frac{\phi(Jt)}{\gamma} = -\frac{\gamma}{4(1+\gamma)} \left(\frac{Jt}{l}\right)^2,$$

$$\phi'(Jt) = -\frac{\gamma}{2(1+\gamma)} \cdot \frac{Jt}{l},$$

or finally
$$\frac{\phi(\zeta)}{\gamma} = -\frac{\gamma}{4(1+\gamma)} \cdot \frac{\zeta^2}{l^2},$$

$$\phi(\zeta) = -\frac{\gamma}{2(1+\gamma)} \cdot \frac{\zeta}{l}.$$

On Plate VIII. Fig. XII. (α) the values of $\phi(\zeta)/\gamma$ are plotted for $\gamma = 1$, 1/2, 1/10, and 1/100. In addition the parabola $\xi = -\frac{\gamma}{4(1+\gamma)} \cdot \frac{\zeta^2}{l^2}$, is drawn for $\gamma = 1$. It coincides extremely closely with the curve $\xi = \phi(\zeta)/\gamma$, except for values of ζ/l less than 2·5. The coincidence would be still more marked for smaller values of γ. We may then say that the angle of rotation is still given notwithstanding the elasticity of the axle by a parabolic formula. We can in fact write

$$\psi = \omega t - \frac{Nl}{K} \cdot \frac{\gamma}{4(1+\gamma)} \left\{\left(\frac{Jt - x}{l}\right)^2 + \left(\frac{Jt + x}{l}\right)^2\right\},$$

and obtain for all values of $t > 2{\cdot}5 + x/J$ results giving the angle of torsion sufficiently accurately for all practical purposes. For lesser values of t we must use the numerical results of Table I (b).

Turning to the second figure of Plate VIII., we see that $\xi = \phi'(\zeta)$ differs considerably from $\xi = -\frac{\gamma}{2(1+\gamma)} \cdot \frac{\zeta}{l}$ (the dotted lines). It is true that the curve system has the same general direction as the line system, but it varies far more than in the case of $\phi(\zeta)$ and the corresponding parabolic system. The chief variation is indeed the principal harmonic, but the others have for large values of γ some influence. It would be easy to deduce a formula giving the effect of the principal harmonic in addition to the linear term, but the Tables and Diagrams will be sufficient to indicate the general nature and practical magnitude of the strain in the axle. As we have in the case of the stress-couple to take the *difference* of two functions like $\phi'(\zeta)$, we obtain no approximation to the stress-couple changes from the linear approximation.

29. It will be found, however, that an empirical formula for the maximum-stress couple, which occurs at the wheel end of the axle is,—so far as we are justified in deducing one from the small range of values of γ considered—

$$\text{Max. couple} = N \,.\, \frac{2\gamma}{1+\gamma}.$$

Although the stress-couples plotted on Plate X. include only two or three maxima, it is still highly probable that the *maximum-maximorum* is owing to viscosity included in these values.

TABLE OF MAXIMA-COUPLES IN AXLE. WHEEL ENDS.

γ	Empirical Formula	1st Maximum	2nd Maximum	3rd Maximum
1	N	·864666 N $(Jt = 2l)$	·881732 N $(Jt = 4{\cdot}5l)$	·784856 N $(Jt = 8{\cdot}25l)$
$\frac{1}{2}$	·666667 N	·632121 N $(Jt = 3l)$	·551791 N $(Jt = 4l)$	·621454 N $(Jt = 8{\cdot}5l)$
$\frac{1}{10}$	·181818 N	·181269 N $(Jt = 2l)$	·176926 N $(Jt = 6l)$	·168494 N $(Jt = 10l)$
$\frac{1}{100}$	·019802 N	·019801 N $(Jt = 2l)$	Maxima practically equal to these occur periodically at intervals of $4l/J$	
$\frac{1}{1000}$	·001998 N	·001998 N $(Jt = 2l)$		

We see that the formula $M = N \,.\, 2\gamma/1+\gamma$ gives for all practical purposes, when $\gamma = > 1/10$ (as most frequently occurs), the value of the stress-couple. We are now able to state a formula for the safe dimensions of our axle, the wheels of which are liable to have the brake suddenly applied.

Let $2P$ be the load borne by the axle (as usual beyond its terminals), let N be the couple due to the brake on each wheel. Then the mean shear* due to the load, if a be the area of the cross-section of the axle, $=P/a$, and the shear due to the sudden application of the brake $=\frac{N}{\sqrt{a}}\cdot\frac{2\gamma}{1+\gamma}\cdot\frac{1}{\gamma' a}$, γ' being the coefficient of equation (v), Art. 4. Hence if S_0 be the safe shearing stress of the material, we must have:

$$P+\frac{N}{\sqrt{a}}\cdot\frac{2\gamma}{(1+\gamma)\gamma'}<aS_0.$$

This formula, as well as the tabular results and the diagrams, apply also to the important practical cases of a propeller shaft or run of shafting terminating in a rotating mass when the motive power is suddenly withdrawn and the system allowed to come to rest by doing work against a constant-couple applied to the rotating mass. In this case P will generally be due to the weight of the rotating mass (*e.g.* propeller, pulley, wheel, etc.). Accordingly in the following remarks 'axle' may be generalised into 'shaft' and 'brake' into resistance-couple.

30. With regard to the general results of Table I (*b*) and Plates VIII. to X. we may note:

(i) That on the application of the brake the uniform retardation of the spin is slightly varied owing to the elasticity of the axle. The maximum variation occurs, as we should naturally expect, at the centre of the axle. See Plate IX. Fig. XIII. (*a*).

(ii) The stress produced by sudden braking is very sensible and may for $\gamma=1$ nearly reach the value produced by applying the brake-couple in statical torsion to the axle. See Plate X. Fig. XIV. (*d*).

(iii) For points between the centre of the axle and 3/8 the length of the axle on either side (*i.e.* the mid three-quarters), the stress in the axle is alternately negative and positive. The positive values of the stress-couple are very sensible at

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* The result given here as in the numerical examples which follow takes account solely of *shearing* stress; to obtain the total stress the traction due to the flexure of the axle would have to be combined with this shear. If ϵ be the distance from the end of the axle to the point of the journal to which load P may be supposed to be applied, the axle, radius r, will be bent by a couple $P\epsilon$, and the maximum traction will be

$$T=\frac{P}{a}\cdot\frac{\epsilon r}{k^2}.$$

This occurs at all points on the topmost fibre of the axle throughout its length. If

$$S=\frac{P}{a}+\frac{N}{a}\cdot\frac{2\gamma}{1+\gamma}\cdot\frac{1}{\gamma'\sqrt{a}},$$

we have to combine T and S. This can be done by de Saint-Venant's radical formula; or very approximately the maximum traction T' (no longer longitudinal), for an isotropic uniconstant material will be, neglecting shear due to P:

$$T'=\frac{1}{2ar}\{3P\epsilon+5\sqrt{P^2\epsilon^2+\tfrac{1}{4}S^2a^2r^2}\}.$$

(See de St Venant's *Leçons de Navier*, p. 416.)

the centre of the axle, and if we accepted Wöhler's conclusions as to alternating stress wearing out material, it might perhaps be necessary for certain axles to consider fully the stresses at points near the centre.

(iv) It is interesting to note that the maxima and minima follow each other with much greater rapidity in the case of an axle to which the brakes are suddenly applied, than in the case of a shaft carrying a pulley thrown suddenly into gear.

31. *Applications**.

I. Each wheel of a truck has to bear a load of 5000 *lbs., the axles are made of wrought iron, the safe shearing stress of which is taken at* 20000 *lbs. per sq. in. Supposing the diameter of each wheel to be* 36″ *and the brakes applied at its circumference, while the axle has a cross-section of* 4 *sq. inches; what is the limit to the safe 'braking' couple which can be applied?*

Instead of taking $P = 5000$ lbs. in the formula of Art. 29, it seems more rational, since the stress is *kinetic*, to take it according to Young's Theorem = 10000 lbs.

We have then as a necessary condition

$$10000 + \frac{N}{2} \cdot \frac{2\gamma}{(1+\gamma)\gamma} < 80000,$$

$$N < \cdot 2821 \times 70000 \times (1 + 1/\gamma), \quad N < 19747\,(1 + 1/\gamma) \text{ inch lbs. of moment.}$$

Or force at circumference $< 1097\,(1 + 1/\gamma)$ lbs.

In most practical cases γ will be much less than 1/10 or we must have

$$\text{Force} < 12067 \text{ lbs.}$$

I have chosen here for the elements values very unfavourable for the safety of the axle, namely a very small cross-section and a practically large value of γ.

If we take braking forces of 500 to 2000 lbs., it would appear that we are well within the limits of safety. At the same time the sudden application of continuous brakes might possibly give rise to much greater values of the 'braking' force on the wheels, and in a heavily laden train (*i.e.* P great) bring the stress dangerously near the limit of safety. To investigate this more closely I take the following example of a "theoretical train."

II. A train is carried on 30 *equal wheels, the load is supposed uniformly distributed and to be* 240 *tons; when the speed of the train is a mile per minute the continuous brake is applied and brings the train to rest in* 200 *yards. To find the nature of the shearing stress in the axles.*

I will suppose the 'braking' force to be 1/2 ton on either side of the wheel, and the friction produced between the metals and wheels when the brake is applied to be constant up to rest. The 'principle of work' gives us at once, if p be the friction at each wheel in tons:

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* See footnote on the preceding page.

$$240\times(1760\times 3)^2=2\times 30p\times 32\times 60\times 60\times 600,$$

whence we have

$$p=1{\cdot}613 \text{ tons.}$$

The total couple applied to each wheel is thus:

$$N=\left(\frac{1}{2}+\frac{1}{2}+1{\cdot}613\right)\times\frac{25}{2}\text{ inch tons}$$

$$=32{\cdot}66 \text{ inch tons,}$$

if 25″ be the diameter of the wheels.

We have next to calculate the value of γ and I shall suppose the axle and wheels to have the following dimensions. *Axle* length 4 feet, diameter 4″; the *wheels* to be treated as belts of metal, 25″ outer diameter, 24″ inner diameter, 4″ broad; both axle and wheels to be of same material. Then the ratio γ of the moments of inertia of half axle and one wheel is given by:

$$\gamma=\frac{2\times 12\times\pi\times 4\times 4/2}{4\times\pi\left(\dfrac{25^2-24^2}{4}\right)\left(\dfrac{25^2+24^2}{8}\right)}={\cdot}0261.$$

We are now able to calculate the value of $\dfrac{N}{\sqrt{a}}\cdot\dfrac{2\gamma}{(1+\gamma)\gamma'}$

$$=\frac{32{\cdot}667}{\sqrt{\pi\times 4}}\times\frac{{\cdot}0522}{1{\cdot}0261\times{\cdot}2821}=1{\cdot}64\text{ tons about.}$$

Thus the maximum shearing stress produced by the sudden application of the brakes $=\dfrac{1{\cdot}64}{a}=\dfrac{1{\cdot}64}{4\pi}$ tons per sq. inch—when we consider torsion only. This occurs anywhere on the circumference of the axle at wheel end.

But there are further stresses which ought to be considered:

(i) a mean vertical uniform shearing stress due to the weight of the train $=8/4\pi$ tons per sq. in.

(ii) a mean horizontal uniform shearing stress due to the friction between the metals and the wheels $=1{\cdot}613/4\pi$ tons per sq. inch.

These give a resultant stress $=8{\cdot}16/4\pi$ nearly in a direction making a slope of ·202 with the vertical.

Because the axle is in rotation, it seems to me proper to double this shearing stress for its kinematic value, or to take it:

$$=\frac{16{\cdot}32}{4\pi}\text{ tons per sq. inch.}$$

Thus the total shearing stress amounts to $17{\cdot}96/4\pi$ tons per sq. in. or say $18/4\pi$ tons per sq. in.

The stress due to the sudden application of the brakes thus consists of two parts,

(i) That due to the sudden frictional resistance of the metals,

(ii) That due to the sudden torsional couple. The latter in this case forms about 9 per cent. of the total shearing stress. Supposing the safe shearing stress (elastic limit) to be about 10 tons, we see that in this case there is a considerable margin of safety.

32. *Case* (x). *The end of a length l of shafting capable of revolving freely is set gradually in motion, so that at the end of an interval τ, it is revolving with speed ω, which is maintained after this interval: to investigate the nature of the stress.*

My object in dealing with this problem is to mark by a particular example, how the interval which a shaft takes to acquire its maximum speed affects the intensity of the stress. We have seen in *Case* (*viii*) (i), that if the end of a shaft be set suddenly in motion with a spin ω, it has to sustain a couple $=K\omega/J$ which gives a remarkably low limit to the spin ω. On the other hand, the lowness of this limit need not be taken to invalidate our theory, for the end of no shaft can be instantaneously given a definite spin; it takes some time, however short, to generate a finite spin in all practical cases. That sudden and rapid rotations may injuriously affect a shaft is evidenced in certain rolling mills where the motion is very rapidly generated and where a special axle piece is put in with the view of its being destroyed and rapidly replaced.

Since our shaft is to rotate freely we have on measuring x from the end set in motion:

$$\psi = f(Jt-x)+f(Jt-2l+x)\text{..........................(i).}$$

Now our conditions for the end set in motion are the following:

$$\frac{d\psi}{dt}=\omega t/\tau \text{ from } t=0 \text{ to } t=\tau\text{...(ii)}$$

$$=\omega \text{ from and after } t=\tau \text{ ....(iii),}$$

$$\left.\begin{array}{l}\dfrac{d\psi}{dx}=0, \text{ when } t=0 \text{ and for all negative value of } t \\ \qquad\qquad \text{for } x=0 \text{ to } l\end{array}\right\}\text{.....................(iv).}$$

We have therefore from (iv)

$$f\left(\zeta \leqq 0\right)=0,$$

$$f'\left(\zeta \leqq 0\right)=0.$$

Let us first deal with $t<\tau$, then we find from (ii)

$$\frac{\omega t}{\tau}=Jf'(Jt)+Jf'(Jt-2l),$$

or if $Jt = \zeta$

$$\frac{\omega}{J} \cdot \frac{\zeta}{J\tau} = f'(\zeta) + f'(\zeta - 2l) \text{.............................(v).}$$

Equation (v) enables us to write down the successive values of $f'(\zeta)$, we have:

$$(a) \qquad f'(\zeta = \text{or} < 0) = 0,$$

$$(b) \qquad f'\left(\zeta = \genfrac{}{}{0pt}{}{2l}{0}\right) = \frac{\omega}{J} \cdot \frac{\zeta}{J\tau},$$

$$(c) \qquad f'\left(\zeta = \genfrac{}{}{0pt}{}{4l}{2l}\right) = \frac{\omega}{J} \cdot \frac{2l}{J\tau},$$

$$(d) \qquad f'\left(\zeta = \genfrac{}{}{0pt}{}{6l}{4l}\right) = \frac{\omega}{J} \cdot \frac{\zeta - 2l}{J\tau},$$

$$(e) \qquad f'\left(\zeta = \genfrac{}{}{0pt}{}{8l}{6l}\right) = \frac{\omega}{J} \cdot \frac{4l}{J\tau},$$

and generally

$$(f) \qquad f'\left(\zeta = \genfrac{}{}{0pt}{}{2 \,.\, 2rl}{2 \,.\, (2r-1)l}\right) = \frac{\omega}{J} \cdot \frac{2rl}{J\tau},$$

$$(g) \qquad f'\left(\zeta = \genfrac{}{}{0pt}{}{2 \,.\, (2r+1)l}{2 \,.\, 2rl}\right) = \frac{\omega}{J} \cdot \frac{\zeta - 2rl}{J\tau}.$$

This continues until $\zeta = J\tau < 2(2r-1)l$ or $< 2 \,.\, 2r \,.\, l$.

I now propose to write down the values of the stress couple $K\, d\psi/dx$. We have:

$$\frac{d\psi}{dx} = -f'(Jt - x) + f'(Jt - 2l + x).$$

$$(a) \qquad Jt < x, \quad \frac{d\psi}{dx} = 0,$$

$$(b) \qquad Jt > x < (2l - x), \qquad \frac{d\psi}{dx} = -\frac{\omega}{J} \cdot \frac{(Jt - x)}{J\tau},$$

$$(c) \qquad Jt > (2l - x) < (2l + x), \quad \frac{d\psi}{dx} = -\frac{\omega}{J} \cdot \frac{2(l - x)}{J\tau},$$

$$(d) \qquad Jt > (2l + x) < (4l - x), \quad \frac{d\psi}{dx} = \frac{\omega}{J} \cdot \frac{Jt - 4l + x}{J\tau},$$

$$(e) \qquad Jt > (4l - x) < (4l + x), \quad \frac{d\psi}{dx} = 0,$$

$$(f) \qquad Jt > (4l + x) < (6l - x), \quad \frac{d\psi}{dx} = -\frac{\omega}{J} \cdot \frac{Jt - 4l - x}{J\tau},$$

$$(g) \qquad Jt > (6l - x) < (6l + x), \quad \frac{d\psi}{dx} = -\frac{\omega}{J} \cdot \frac{2(l - x)}{J\tau},$$

$$(h) \qquad Jt > (6l + x) < (8l - x), \quad \frac{d\psi}{dx} = \frac{\omega}{J} \cdot \frac{Jt - 8l + x}{J\tau},$$

(*i*) $Jt > (8l - x) < (8l + x), \quad \dfrac{d\psi}{dx} = 0,$

(*j*) $Jt > (8l + x) < (10l - x), \quad \dfrac{d\psi}{dx} = -\dfrac{\omega}{J} \cdot \dfrac{Jt - 8l - x}{J\tau}.$

The law of succession is now sufficiently obvious and is represented in Fig. VI. (*a*), Plate II. *We note that (if $J\tau > 2l$), the maximum stress-couple always takes place at the rotated terminal and has for values* $\pm \dfrac{K\omega}{J} \cdot \dfrac{2l}{J\tau}$.

I append, to shew how the shaft assumes the spin ω, a diagram of the values of $d\psi/dt$ for $Jt < J\tau$ (Fig. VI. (*b*), Plate II.).

We see from these diagrams that although the stresses take a considerable range of values, yet that the spin throughout the rod is always of the same sense and steadily increases, and that while occasionally the same throughout the rod, it never differs by more than $\omega l/J\tau$ at any two points at the same instant, and these two points are then the two terminals or those farthest apart. We easily see that, if $J\tau$ is considerable as compared with l, the spin becomes practically uniform throughout the shaft by the time $t = \tau$; further, that if $J\tau$ be great as compared with $2l$, the couple becomes practically zero before the rotation is brought up to its maximum ω.

On the other hand we observe that if $J\tau = 4rl$ (r being an integer), whatever τ be, then the spin will be uniform along the shaft and the couple zero throughout when the maximum spin is reached.

We have now to investigate whether any peculiarity occurs in the stress for values of $t > \tau$, *i.e.* after the terminal spin becomes uniform and equal to ω.

Now $$\omega = \omega \frac{t}{\tau} - \frac{\omega (t - \tau)}{\tau},$$

or, it may be considered as the result of superposing an opposite gradual spin $-\omega (t - \tau)/\tau$. Commencing at the end of the interval τ, upon the already existing spin $\omega t/\tau$, we have then only to superpose the diagram of the previous article for $d\psi/dx$ shifted through a horizontal distance $J\tau$ upon itself, and (ω being changed to $-\omega$ and t to $t - \tau$ in the shifted diagram) simple addition gives the required succession of stress as in the cases of repeated loading we have previously dealt with.

Evidently we must have either

(i) $J\tau = 2r \,.\, 2l + J\tau'$, where $J\tau' < 2l$,

or (ii) $J\tau = \overline{2r+1} \,.\, 2l + J\tau'$, where $J\tau' < 2l$.

In the former case, take $Jt = 2r \,.\, 2l + Jt'$,

in the latter $Jt = \overline{2r+1} \,.\, 2l + Jt'$;

or we transfer the epoch nearer to the interval $t = \tau$. As we are only concerned with what takes place after $t = \tau$, we shall deal only with that portion of the diagram for which $Jt = 2r \,.\, 2l$ or $\overline{2r+1} \,.\, 2l$ or greater values.

Case (*i*). $J\tau = 2r \,.\, 2l + J\tau'$.

The stress immediately following $2r \,.\, 2l$ now takes the form $-\dfrac{\omega}{J} \cdot \dfrac{Jt' - x}{J\tau}$, or we may represent it by the original diagram starting from the origin with Jt' put for Jt; the diagram to be superposed is also the original diagram starting from the origin, with $-\omega$ for ω and $Jt' - J\tau$ for Jt; the shift is to be something less than $2l$. Fig. VII. (*a*), Plate II. gives the resulting system of stress. We notice at once that the maximum positive and negative stress-couples are given by

$$\pm K \cdot \frac{\omega}{J} \cdot \frac{J\tau'}{J\tau}.$$

Now except in the case where the duration of the load is less than $2l/J$, these are always less than $\pm K \dfrac{\omega}{J} \cdot \dfrac{2l}{J\tau}$, hence the maximum stress is reached before the maximum spin at the rotated terminal and equals $K \dfrac{\omega}{J} \cdot \dfrac{2l}{J\tau}$.

If the wave has run a considerable number of times along the shaft, before the maximum spin is reached, *i.e.* $2r =$ large number, the stresses afterwards are less than $K \dfrac{\omega}{J} \cdot \dfrac{1}{2r}$ and are therefore very small. The maximum stresses occur at the rotated end.

If $2r = 0$, or $J\tau$ be $< 2l$, then $J\tau' = J\tau$ and the maximum stress couples now become $\pm K\omega/J$ and take place after the instant at which the maximum spin is reached. Obviously then the danger is much increased if the maximum spin is attained by the rotated end before the head of the wave has returned to that end. The practical rule then is that long shafts which are rapidly rotated should be set very gradually in motion.

Case (*ii*). $J\tau = \overline{2r + 1} \,.\, 2l + J\tau'$.

In this case we note, see Fig. VII. (*b*), that the maximum stress-couples again occur at the rotated end, but that their values are now

$$\pm \frac{\omega}{J} \cdot \frac{J\tau'}{J\tau}, \qquad \pm \frac{\omega}{J} \cdot \frac{J\tau' - 4l}{J\tau} \qquad (J\tau' < 2l).$$

Hence the second pair except when $J\tau'$ just equals $2l$ always gives the maximum stress-couple, *i.e.* $\pm K \dfrac{\omega}{J} \cdot \dfrac{4l - J\tau'}{J\tau}$. If $J\tau' = 0$, $J\tau = 2l$, the shaft has to withstand the large couple $\pm K \,.\, 2\omega/J$ which follows immediately on the maximum spin being attained by the rotated end. Thus the most dangerous manner in which a spin ω can be given to a shaft is applying it gradually in an interval $= 2l/J$; in this case the couple amounts to *double the maximum which would be reached by an instantaneous application of the spin.*

Whenever the period up to maximum rotation $= \overline{2r + 1} \,.\, 2l/J$ then the maximum couple is $K\frac{\omega}{J}\cdot\frac{2}{2r+1}$ and we note that it is very much greater than the value $K\frac{\omega}{J}\cdot\frac{1}{2r}$ obtained for a period $=2r\,.\,2l/J$.

Of course if the period be great (*i.e.* r great) these couples may be made very small and so quite negligible; when, however, the period of getting up spin is only a few times $2l/J$, the period for sound running along the shaft and back again, then the stress-couples produced solely by the setting in motion must be taken into account, and it will be important to ascertain whether τ lies between $\overline{2r-1}\,.\,2l/J$ and $2r\,.\,2l/J$ or between $2r\,.\,2l/J$ and $\overline{2r+1}\,.\,2l/J$.

33. We may illustrate the preceding results by the following numerical example. What speed can be safely generated in 1·5 seconds in a shaft 10 ft. long and 2·5″ diameter, the material being wrought iron?

In this case taking $J=10000$ ft. per sec. we have $J\tau=15000$ feet $=750\times 2l$. This falls under the first case $J\tau=2r\,.\,2l+J\tau'$. Hence the maximum stress-couple

$$=\frac{K\omega}{J}\cdot\frac{2l}{J\tau}=\frac{1}{750}\cdot\frac{K\omega}{J},$$

or only 1/750 of the stress produced by a sudden throwing into gear.

Suppose, however, the time of generating the speed had been 1·502 secs., then we have $J\tau=15020$ feet $=751\times 2l$. This result now falls under Case (ii) and we have the maximum stress-couple

$$=\frac{K\omega}{J}\cdot\frac{4l}{J\tau}=\frac{1}{375\cdot 5}\cdot\frac{K\omega}{J},$$

that is practically double of the value in the previous case. As it would be impossible in practice to determine whether the speed was generated in 1·5 or 1·502 seconds, it is obvious that the latter value is to be taken for safety. The result shews how the maximum stress varies largely with extremely slight differences in the time of getting up speed; a *slower* generation corresponding here to a *greater* stress.

By Art. (4), the maximum couple a circular shaft will sustain without injury is:

$$M_0=\cdot 2821\times S_0\sqrt{a^3}\ \text{and}\ K=\mu\times a\times\frac{(2\cdot 5)^2}{8}.$$

For wrought iron we may take $\mu=10{,}500{,}000$ and $S_0=20{,}000$,

$$\therefore\ 10{,}500{,}000\times a\times\frac{(2\cdot 5)^2}{8}\times\frac{\omega}{J}\times\frac{1}{375\cdot 5}=\cdot 2821\times 20{,}000\times a\times\sqrt{\frac{\pi}{4}(2\cdot 5)^2},$$

or $\omega=$ about 10928 revolutions per sec.

Thus the safe speed is increased from about 58 to 21856 revolutions per sec. in a shaft of these dimensions by spreading the generation of speed over an interval of 1·5 secs. If the duration of getting up speed were only so small as 1/100 of a second, still a speed of somewhere about 75 revolutions per sec. might be safely generated in this time. A very small time thus suffices to allow of any speed likely to occur in practice to be safely generated.

One point deserves to be noticed: *For the same length of shafting the safe number of rotations per second varies inversely as the diameter, the time of getting up speed being the same.* Hence a shaft of smaller diameter might withstand better than one of larger a sudden start. Of course the destruction will depend not only on the effect of a sudden start, but also on the magnitude of the resistance it has to overcome. If, however, a shaft liable to sudden throwing in and out of gear, can withstand the torsional resistance, then the smaller the diameter the greater the safety. It does not seem improbable that there may exist practical cases in which a shaft of smaller diameter would have a longer life than one of larger, namely cases in which the impulsive action due to sudden alteration of speed, or even its reversal, is a large factor in the production of stress, (*e.g.* in the small bits of axle inserted next the rollers in rolling mills with the express object that the shaft, which soon gets worn out, shall rupture at this point and not elsewhere).

34. All the solutions we have yet given to the problem of rotating shafting are in a certain sense incomplete; we have only calculated the value of the stress up to a certain time, and not its general value at any time after the impulse. This result is largely due to the nature of the arbitrary functions we have used: the differential-finite-difference equation on which their discovery depended did not give us any general solution. This limitation of time is of no importance in practical applications where the important point is to determine the maximum stress, which, even disregarding the effect of viscosity, is reached very shortly after the impulse and within a few passages of the wave backwards and forwards along the shaft. In order, however, to point out one or two features of the solutions we have been considering, I propose to discuss *Case (vii)* from the standpoint of that method of investigation which uses 'normal coordinates.'

35. *Case (vii bis). One end of a length l of shafting is maintained at a constant spin ω; at the other end of this shafting is a wheel of moment of inertia $m\beta^2$, which together with the shafting is initially rotating with the same spin ω. Suddenly a frictional couple N is applied to the wheel: To determine the impulsive stress in the shafting.*

As in Art. 17, we have the system of equations

$$\frac{d^2\psi}{dt^2} = J^2 \frac{d^2\psi}{dx^2} \quad \text{..........} \quad \text{(i)},$$

$$m\beta^2 \frac{d^2\psi}{dt^2} = K\frac{d\psi}{dx} - N, \text{ when } x = 0, \text{ for all values of } t > 0 \quad \text{..........} \quad \text{(ii)},$$

$$\frac{d\psi}{dt}=\omega, \text{ when } x=l, \text{ for all values of } t \quad\text{......................(iii)},$$

$$\frac{d\psi}{dx}=0, \text{ when } t<0, \text{ for all values of } x \text{ from } 0 \text{ to } l \quad\text{........(iv)},$$

$$\frac{d\psi}{dt}=\omega \quad\text{..(v)}.$$

As a second method of solution let us find a trigonometrical series satisfying this system. Assume

$$\psi=\omega t+\frac{N}{K}(x-l)+\chi \quad\text{..(vi)}.$$

Thus we have

$$\frac{d^2\chi}{dt^2}=J^2\frac{d^2\chi}{dx^2} \quad\text{..(i')},$$

$$m\beta^2\frac{d^2\chi}{dt^2}=K\frac{d\chi}{dx}, \text{ when } x=0 \quad\text{..(ii')},$$

$$\frac{d\chi}{dt}=0, \text{ when } x=l, \text{ for all values of } t \quad\text{......................(iii')},$$

$$d\chi/dx=-N/K, \text{ when } t=0, \text{ for all values of } x \quad\text{.................(iv')},$$

$$\frac{d\chi}{dt}=0, \text{ when } t=0, \text{ for all values of } x \quad\text{......................(v')}.$$

The value of χ satisfying (i′), (iii′) and (v′) is

$$\chi=\Sigma A_n \sin n(l-x)\cos nJt,$$

where we must determine n from (ii′) and A_n from (iv′).

Substituting in (ii′) we have

$$-m\beta^2J^2n\sin nl=-K\cos nl,$$

or

$$nl\tan nl=lK/(m\beta^2J^2)=\gamma \quad\text{..........................(vii)},$$

where γ has the same value as in Art. 14 (p. 20). The values of nl are thus the roots of the transcendental equation (vii).

Finally to find A_n we have from (iv′)

$$N/K=\Sigma A_n n\cos n(l-x).$$

By means of (vii) we can easily shew, if n_1 and n_2 be unequal, that

$$\int_0^l \cos n_1(l-x)\cos n_2(l-x)\,dx=0,$$

whence it follows that

$$\frac{N}{K}\int_0^l \cos n(l-x)\,dx=A_n n\int_0^l \cos^2 n(l-x)\,dx,$$

or

$$A_n=\frac{2Nl}{K}\cdot\frac{\sin nl}{nl(nl+\sin nl\cos nl)} \quad\text{......................(viii)}.$$

The full solution is thus comprised in

$$\left.\begin{array}{l}\psi=\omega t+\dfrac{N}{K}(x-l)+\Sigma\dfrac{2Nl}{K}\cdot\dfrac{\sin nl}{nl\,(nl+\sin nl\cos nl)}\sin n\,(l-x)\cos nJt,\\ \text{where the summation extends to all values of } nl \text{ which are roots of}\\ \qquad\qquad nl\tan nl=\gamma\end{array}\right\}\text{......(ix).}$$

The equation (ix) gives theoretically the complete analytical solution, but it is obviously of very little value for practical purposes. For every value of γ we should have to calculate the roots of the transcendental equation afresh, and having found these roots substitute in the trigonometrical series and calculate its terms for each separate value of x and Jt. The labour involved in this process would be very great, and without taking a very large number of terms the solution would be nothing like so approximate as the calculations from the arbitrary functions.

36. The only case where I am aware of a similar calculation having been made is that of a beam terminally supported and struck at its centre. This was dealt with by de Saint-Venant in 1853—4, when a paper was presented to the *Société philomathique* followed by a paper in 1857 to the *Académie*. (The numerical results are given on pp. 553—563, with several corrections at the end of the volume, in the annotated *Clebsch*). The curves for the displacement of the beam at successive intervals drawn by de Saint-Venant have never been published and will not now be, as he himself felt that they had not been calculated for a sufficiently wide range of the ratio of the mass of the beam to the mass of the striking body, nor again for a sufficient number of points along the beam. They are represented so far as de Saint-Venant's calculations permit in his plaster of Paris model of impact on a beam.

Bearing in mind the great labour de Saint-Venant devoted to the particular case of transverse impulse treated by him, it seemed obvious that for practical purposes the numerical calculations must be based on the more readily dealt with arbitrary functions. These arbitrary functions express in fact the sums of the two trigonometrical series obtained by splitting up the terms such as $\sin n\,(l-x)\cos n\,Jt$ into the sum of two sines. Various interesting identities may thus be obtained concerning series in $\sin nx$ and $\cos nx$, where n is a root of the transcendental equation $nl\tan nl=\gamma$, but their consideration would carry us beyond the limits of our present topic.

One particular case may be dealt with by the trigonometrical method as a means of verifying our previous results. Let us suppose γ small, or the moment of inertia of the wheel large as compared with that of the shaft, we may then throw the solution (ix) into a simpler form. There are two distinct kinds of solutions in this case for the transcendental equation $nl\tan nl=\gamma$.

All the roots but one may be expressed in integer powers of γ. I find approximately for them if i be any integer:

$$nl = i\pi + \frac{\gamma}{i\pi} - \frac{\gamma^2}{i^3\pi^3} - \frac{\gamma^3}{i^3\pi^3}\left(\frac{1}{3} - \frac{2}{i^2\pi^2}\right) + \frac{\gamma^4}{i^5\pi^5}\left(\frac{4}{3} - \frac{5}{i^2\pi^2}\right) \text{............(x)},$$

which converges rapidly even for not very small values of γ. The remaining root $n_0 l$ depends upon powers of $\sqrt{\gamma}$ and I find approximately

$$n_0 l = \gamma^{\frac{1}{2}} - \frac{1}{6}\gamma^{\frac{3}{2}} + \frac{11}{360}\gamma^{\frac{5}{2}} - \frac{17}{5040}\gamma^{\frac{7}{2}} \text{........................(xi)}.$$

It remains to substitute these values in the expression for A_n. For purposes of comparison, I will calculate the quantities involving A_n which occur in the values of the stresses at both terminals of the shaft. We have

$$K\frac{d\psi}{dx} = N - 2N\Sigma\,\frac{\sin nl}{nl + \sin nl \cos nl}\,.\cos n\,(l - x)\cos n\,Jt.$$

Hence $x = l$

$$M_l = N\left\{1 - 2\Sigma\,\frac{\sin nl}{nl + \sin nl \cos nl}\cos nl\frac{Jt}{l}\right\} \text{......................(xii)}$$

and $x = 0$
$$M_0 = N\left\{1 - \Sigma\,\frac{\sin 2nl}{nl + \sin nl \cos nl}\,.\cos nl\frac{Jt}{l}\right\} \text{......................(xiii)}.$$

Further we have

$$\frac{d\psi}{dt}_{(x=0)} = \omega - \frac{2JN}{K}\,\Sigma\,\frac{\sin^2 nl}{nl + \sin nl \cos nl}\sin nl\frac{Jt}{l} \text{............(xiv)}.$$

37. In the following tables I have calculated (i) the values of the first five nl's for $\gamma = 1$, $1/2$, $1/4$, $1/6$, $1/10$; Table II. (ii) the corresponding values of the coefficients in equations (xii), (xiii) and (xiv); Table III. In Table II. the first values are radians, the second are degrees and minutes.

TABLE II*. *For large values of γ.*

γ	$n_0 l$	$n_1 l$	$n_2 l$	$n_3 l$	$n_4 l$
1	·86035 49° 17′ 40″	3·42562 196° 16′ 25″	6·43731 368° 49′ 50″	9·52933 545° 59′ 26″	12·64529 724° 31′ 17″
$\frac{1}{2}$	·65330 37° 25′ 53″	3·29231 188° 38′ 8″	6·36162 364° 29′ 38″	9·47749 543° 1′ 12″	12·60601 722° 16′ 17″
$\frac{1}{4}$	·48009 27° 30′ 27″	3·21910 184° 26′ 27″	6·32271 362° 15′ 52″	9·45122 541° 30′ 54″	12·58623 721° 8′ 17″
$\frac{1}{6}$	·39725 22° 45′ 38″	3·19373 182° 59′ 15″	6·30959 361° 30′ 47″	9·44243 541° 0′ 40″	12·57962 720° 45′ 32″
$\frac{1}{10}$	·31105 17° 49′ 19″	3·17310 181° 48′ 18″	6·29906 360° 54′ 34″	9·43538 540° 36′ 26″	12·57432 720° 27′ 20″

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* The values here given have been obtained from those of Equations (x) and (xi) as *first* approximations. I believe the radians correct to the last place of decimals, as they were calculated

TABLE III. *For large values of* γ.

γ		n_0l	n_1l	n_2l	n_3l	n_4l
1	c_1	1·11914	−·15169	·04658	−·02167	·01239
	c_2	·72986	·14561	·04604	·02155	·01235
	c_3	·33695	·02125	·00357	·00113	·00049
$\frac{1}{2}$	c_1	1·07014	−·08728	·02433	−·01106	·00627
	c_2	·84973	·08629	·02426	·01106	·00626
	c_3	·32522	·00655	·00095	·00029	·00012
$\frac{1}{4}$	c_1	1·03820	−·04697	·01242	−·00558	·00315
	c_2	·92083	·04684	·01241	·00558	·00315
	c_3	·23975	·00182	·00025	·00007	·00003
$\frac{1}{6}$	c_1	1·02620	−·03213	·00833	−·00373	·00210
	c_2	·94629	·03207	·00833	·00373	·00210
	c_3	·19851	·00084	·00011	·00003	·00001
$\frac{1}{10}$	c_1	1·01609	−·01966	·00503	−·00224	·00126
	c_2	·96734	·01965	·00503	·00224	·00126
	c_3	·15559	·00031	·00004	·00001	·00000(5)

$$c_1{}^* = \frac{4\sin nl}{2nl + \sin 2nl}, \qquad c_2 = \frac{2\sin 2nl}{2nl + \sin 2nl}, \qquad c_3 = \frac{2\sin^2 nl}{2nl + \sin 2nl}$$

to 7 places of decimals, the values in degrees, minutes, and seconds were calculated from the values of nl to 7 places; a second value differing by 5″ was then calculated, both were substituted in the equation $nl \tan nl = \gamma$ and a value interpolated between them by a simple proportion to make the right-hand side exactly γ. In nearly all cases when substituted this value gave the right hand a value differing from γ only after the fifth place of decimals. The degree values of nl are thus in a slight degree more approximate than the radian values. I do not think the maximum error can be more than ±3″, and this probably only in the column for n_1l.

* The approximate values of c_1, c_2, c_3 to the third power of γ are

$$\left.\begin{aligned} c_1 &= \frac{(-1)^i 2\gamma}{i^2\pi^2}\left\{1 - \frac{3\gamma}{i^2\pi^2} - \frac{\gamma^2}{i^2\pi^2}\left(\frac{1}{2} - \frac{10}{i^2\pi^2}\right)\right\} \\ c_2 &= \frac{2\gamma}{i^2\pi^2}\left\{1 - \frac{3\gamma}{i^2\pi^2} - \frac{\gamma^2}{i^2\pi^2}\left(1 - \frac{10}{i^2\pi^2}\right)\right\} \\ c_3 &= \frac{\gamma^2}{i^3\pi^3}\left\{1 - \frac{4\gamma}{i^2\pi^2}\right\} \end{aligned}\right\} \quad \text{..............................(xv).}$$

I have used these however only for purposes of comparison and verification and calculated the c's from their transcendental values by substituting in them the values of nl given in Table II.

For all *technical* (as distinguished from physical) problems Table III gives sufficient coefficients for calculating the values of the maximum stress and of the variations of the spin, except perhaps for the case of $\gamma=1$. If further values of the coefficients for this case are required they may be calculated with sufficient approximation from the formulae

$$c_1=(-1)^i\left\{\frac{2}{i^2\pi^2}-\frac{7}{i^4\pi^4}\right\},\qquad c_2=\frac{2}{i^2\pi^2}-\frac{8}{i^4\pi^4},\qquad c_3=\frac{1}{i^3\pi^3}.$$

I add a supplementary table for these additional values:

TABLE IV. *High values of nl for* $\gamma=1$.

$\gamma=1$	n_5l	n_6l	n_7l	n_8l
c_1	-·00798	·00557	-·00411	·00315
c_2	·00794	·00556	·00410	·00315
c_3	·00026	·00015	·00009(54)	·00006

Any calculations made from Tables III and IV for the maximum stress will then have an error of less than 1/2 per cent.

38. The above large values of γ, although theoretically of the greatest interest, are not those which most frequently occur in shafting. Such values are those lying between ·05 and ·005. Accordingly I have added a fourth table, Table V, giving a series of additional small values for γ and the corresponding coefficients for n_0l and n_1l. For practical purposes it is unnecessary to go further, and this table enables us to read off very easily the value of the stress at either end of the shaft and the variation in the spin, due to the suddenly applied frictional resistance. We have for most practical purposes

$$\left.\begin{aligned} M_l &= N\left\{1-c_1\cos n_0l\frac{Jt}{l}-c_1'\cos n_1l\frac{Jt}{l}\right\}\\ M_0 &= N\left\{1-c_2\cos n_0l\frac{Jt}{l}-c_2'\cos n_1l\frac{Jt}{l}\right\}\\ \left(\frac{d\psi}{dt}\right)_{x=0} &= \omega-\frac{2JN}{K}c_3\sin n_0l\frac{Jt}{l}\end{aligned}\right\}\text{..................(xvi)},$$

$x=0$ being the terminal bearing the wheel at which the frictional couple N is applied.

TABLE V*. *For small values of* γ.

γ	n_0l	c_1	c_2	c_3	n_1l	c_1'	c_2'
·08	·27912 15° 59′ 33″	1·01296	·97376	·13954	3·16685 181° 26′ 49″	– ·01582	·01582
·05	·22176 12° 42′ 21″	1·00819	·98355	·11087	3·15743 180° 54′ 26″	– ·00998	·00998
·03	·17234 9° 52′ 28″	1·00495	·99005	·08617	3·15111 180° 32′ 43″	– ·00601	·00601
·02	·14095 8° 4′ 33″	1·00332	·99338	·07048	3·14795 180° 21′ 51″	– ·00403	·00403
·015	·12217 6° 59′ 59″	1·00250	·99565	·06109	3·14636 180° 16′ 24″	– ·00303	·00303
·01	·09983 5° 43′ 12″	1·00167	·99667	·04992	3·14477 180° 10′ 56″	– ·00202	·00202
·008	·08932 5° 7′ 4″	1·00133	·99733	·04466	3·14414 180° 8′ 45″	– ·00162	·00162
·005	·07065 4° 2′ 53″	1·00083	·99834	·03533	3·14318 180° 5′ 28″	– ·00101	·00101
·003	·05474 3° 8′ 12″	1·00050	·99900	·02737	3·14255 180° 3′ 17″	– ·00061	·00061
·002	·04471 2° 33′ 41″	1·00033	·99933	·02235	3·14223 180° 2′ 11″	– ·00041	·00041
·001	·03162 1° 48′ 42″	1·00017	·99966	·01581	3·14191 180° 1′ 6″	– ·00020	·00020
·0005	·02236 1° 16′ 52″	1·00008	·99983	·01118	3·14165 180° 0′ 33″	– ·00010	·00010
·0001	·01 0° 34′ 23″	1·00002	·99996	·00499	3·14159 180° 0′ 7″	– ·00002	·00002

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* The values of c_1 or c_2 for n_1l—since they are for the above values of γ equal in magnitude to five places of decimals—have been determined at first by the trigonometrical formula and afterwards by the expression $\cdot 202642 \times \gamma - \cdot 061596 \times \gamma^2$, which was very soon quite sufficient. The values of c_3 for n_1l may be obtained if needed from the expression $\cdot 032252\gamma^2 - \cdot 013072\gamma^3$, but for all practical purposes their magnitudes are negligible. The values of c_1 and c_2 for n_2l may also be readily determined from Equation (xv). But for all technical purposes we may legitimately neglect them as well as the values corresponding to n_1l when they are $< \cdot 03$.

The values of c_1, c_2, c_3 for n_0l have been determined from the trigonometrical expressions by substituting in them the value of n_0l to 7 places of decimals (*i.e.* not from the five places tabulated). This was continued till the following approximate expressions gave correct results to five places of decimals:

$$c_1 = 1 + \frac{\gamma}{6} - \frac{\gamma^2}{360}, \qquad c_2 = 1 - \frac{\gamma}{3} + \frac{16}{45}\gamma^2, \qquad c_3 = \frac{\gamma^{\frac{1}{2}}}{2}\left(1 - \frac{\gamma}{6}\right) \text{ ....(xvii).}$$

39. I propose now to discuss the practical conclusions which may be drawn from these formulae, and to plot out the values of M_l, M_0 and $\left(\frac{d\psi}{dt}\right)_0$ for several values of γ.

(i) For most practical values of γ, c_1' and c_2' may be neglected, hence the maximum-minimum couples at the wheel which gives out and at the wheel which takes in the work are respectively $N(1 \pm c_1)$ and $N(1 \pm c_2)$. Thus we see from the tables that the maximum couple occurs at the end of the shaft which is giving out work and is always greater (although often only slightly) than $2N$. The couple at this end can also change its sign and take a small negative value since $c_1 > 1$.

On the other hand the maximum couple at the work-receiving end is always less than $2N$ since c_2 is < 1 and thus cannot change its sign.

(ii). The maximum variations in the spin ω are $\pm 2 (JN/K) C_3$. For many spins ω and frictional resistances N, these variations will be small, but in other cases it is quite possible that they may be commensurable with ω. It is obvious that they do not depend upon ω, but only on the magnitude of N for the same shaft and pulley. For a circular shaft of diameter $2a$, $K = \mu \alpha k^2$ (if $\alpha = \pi a^2$) and the maximum stress-couple has for its limiting value $M_0 = N\{1 + c_1 + c_1' + c_1'' + \ldots\}$, see Art. 42. But M_0 cannot be greater than $S_0 \alpha k^2/a$. Thus the greatest possible value of the variation $\frac{2JN}{K} \cdot c_3$ is equal to $\frac{2c_3}{1 + \Sigma c_1} \cdot \frac{S_0}{\mu} \cdot \frac{J}{a}$ or to $\frac{c_3}{1 + \Sigma c_1} \cdot \frac{S_0}{\mu} \cdot \frac{J}{a} \cdot \frac{1}{\pi}$ revolutions per second. In the case of wrought iron when we put $\mu = 10{,}500{,}000$, $S_0 = 20{,}000$, $J = 10{,}000 \times 12$ in inches, we have $\frac{c_3}{1 + \Sigma c_1} \cdot \frac{72{\cdot}76}{a}$ revolutions per second, a being read in inches.

Hence, *if the frictional resistance approaches that which would destroy the elasticity of the shaft, it seems possible that, for a large c_3 (i.e. a large γ) and a small a, the spin at the wheel might even become negative.*

Of course it must be remembered that we have taken a very large value for N,—its limiting value. Still it seems important to note the considerable variations in spin which can take place owing to a machine being suddenly

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The approximate expressions (x), (xi), (xv) and (xvii) will enable the reader to calculate by the aid of the following as many further values of nl and c as he may require.

$i=$	$i\pi$	$1/i\pi$	$1/i^2\pi^2$	$1/i^3\pi^3$	$1/i^4\pi^4$	$1/i^6\pi^6$	$\frac{1}{3i^2\pi^2} - \frac{2}{i^5\pi^5}$	$\frac{4}{3i^5\pi^5} - \frac{5}{i^7\pi^7}$
1	3·141593	·318310	·101321	·032252	·010266	·001040	·004215	·002702
2	6·283186	·159155	·025330	·004031	·000642	·000016	·001140	·000121
3	9·424779	·106103	·011258	·001194	·000127	·000001	·000372	·000017
4	12·566372	·079577	·006332	·000504	·000040	·000000	·000162	·000004

thrown in or out of gear in a shafting. One would expect the effect to be distinctly prejudicial in the case of a number of weaving machines on the same shaft, where uniformity of spin is so essential to uniformity of work, that one powerful engine with a long shaft is frequently replaced by a number of small engines each working only a few looms.

For example, suppose N to be only 1/8 of the maximum safe couple, and let γ have the value it has for Case (α) of the following example, then $c_3 = \cdot 233$ about and Σc_1 may be taken as 1·1, $a = 2\cdot 5/2$, then the variations amount to as much as 7·7 revolutions a second, and if the shaft were being driven at the slow rate of 100 revolutions per minute, the effect of throwing a machine into gear might stop and even reverse the spin of the pulley on the shaft. I do not know whether any effects of this kind have been observed, the ordinary values of γ and c_3 in many cases rendering the variations very much smaller as compared with ω. The effect too, of the usually progressive, if very rapid, application of the frictional resistance would tend to diminish the variation, as we have seen in a like case in Art. 33. At the same time it must be remembered that the mean value of the spin even for such a short interval as a second is practically ω and thus positive; the period in such an unfavourable case as (α) of the following article for the variation in the spin being only ·00674 of a second. Thus the negative spin would probably only manifest itself as a scarcely sensible vibration of the pulley.

40. (iii) The point of the above remark will perhaps be brought out more clearly by the following numerical example.

A wrought-iron shaft of 2·5″ diameter runs along the side of a shop 50′ long. One terminal is started with a speed of 100 revolutions a minute. At the other terminal is a cast-iron belt pulley which being thrown into gear is subjected to a frictional couple of 20000 inch lbs. Determine the maximum couples at the terminals of the shaft and the variations in the spin of the pulley when it has the following dimensions:

Case (α)		Case (β)	
	Breadth = 4″		Breadth = 6″
	Thickness of rim = ·445″		Thickness of rim = 1′
	Radius = 10″		Radius = 20″.

We shall neglect the spokes of the pulley, and suppose its inertia practically due to the weight of the rim. The density of wrought iron to that of cast iron will be taken as 480 : 450 or $16 : 15 = (\rho/\rho')$. Dealing first with the pulley (α) we have for its moment of inertia

$$\rho' \frac{\pi}{2} \{10^4 - 9\cdot 555^4\} \times 4 = \rho' \frac{\pi}{2} \times 1664\cdot 67 \times 4,$$

while the moment of inertia of the shaft

$$=\rho \times \pi (1{\cdot}25)^2 \times 600 \times \frac{(1{\cdot}25)^2}{2} = \rho \times \frac{\pi}{2} (1{\cdot}25)^4 \times 600.$$

Hence the ratio $\gamma = {\cdot}2346$ nearly.

In the case of the pulley (β) we have for its moment of inertia

$$\rho' \pi/2 \{(20)^4 - (19)^4\} \times 6,$$

and for γ the value $$\frac{16}{15} \cdot \frac{(1{\cdot}25)^4 \times 600}{761 \times 39 \times 6} = {\cdot}00877.$$

It is thus obvious that not quite doubling the size of the belt pulley enormously decreases the value of γ.

Before seeking for the values of the constants, I will determine the limit to the frictional couple which could be applied statically to a shaft of the above kind. For torsion τ the couple M is given by:

$$M = \mu\tau \times \text{moment of inertia of cross-section.}$$

Hence, if S_0 be the safe shear limit $= \mu\tau \times a$, where a is the radius, we have for the maximum couple:

$$M_0 = \frac{S_0 \pi a^3}{2}.$$

Taking $S_0 = 4/5\, T_0$ and $T_0 = 25000$ lbs. per sq. in. for wrought iron, we have

$$M_0 = 20000 \times 3{\cdot}14159 \times \frac{(1{\cdot}25)^3}{2} = 31415{\cdot}9 \times (1{\cdot}25)^3 = \text{about } 61000 \text{ inch lbs.}$$

The couple on our pulley is only 20000 inch lbs., but it does not follow that we are thus within the safe limits for working stress, *because the maximum couple applied to the wheel does not measure the maximum couple in the shafting*. It is necessary that $N(1+c_1)$ should be less than M_0. This is true in our case because c_1 is only slightly greater than unity, but it would not be true if the frictional couple had been 35000 inch lbs. instead of 20000, a magnitude, however, much below the maximum statical couple 61000 inch lbs. I will now give the values of the constants.

	Case (α)		Case (β)	
	Calculated	Interpolated from tables	Calculated	Interpolated from tables
c_1	1·03604	1·03598	1·001461	1·001459
c_2	·92544	·92552	·997090	·997086
c_3	·23288	·23219	·046756	·046759
c_1'	– ·04429	– ·04423	– ·001773	– ·001774
c_2'	·04416	·04411	·001773	·001774
c_3'	·00162	·00164	·0000025	—

In Case (α) the interpolated values of the constants have been obtained from Table III by taking the difference between 1/4 and 1/6 and using proportional parts. In case (β) the difference was taken as the mean of the differences up and down from the number (·008) to which γ(·00877) was closest, in order to obtain a difference as close to ·008 as possible and then using proportional parts. It will be seen that when γ is only sensible in the third place of decimals ($=\cdot00877$), then the interpolated values of the constants agree to five places of decimals with the calculated and thus there ought to be no hesitation in using in this case the method of interpolation. When γ is sensible in the first place of decimals (·23465) we have to work from Table III, where the differences given for γ are far greater but the maximum error is here only ·0007 (occurring in c_3) which we may fairly neglect for practical purposes.

We have also for Case (α),	for Case (β)
$n_0 l = \cdot 4662673$	$n_0 l = \cdot 0935116$
$= 26^\circ\, 42'\, 55''$	$= 5^\circ\, 21'\, 28''$
$n_1 l = 3\cdot 21446$	$n_1 l = 3\cdot 1443821$
$= 184^\circ\, 11'\, 11''$	$= 180^\circ\, 9'\, 35''.$

Our numerical results are therefore for Case (α)

$$M_l\,(=\text{couple at rotated end}) = N\left\{1 - 1\cdot 03604 \cos n_0 l \frac{Jt}{l} + \cdot 04429 \cos n_1 l \frac{Jt}{l}\right\},$$

$$M_0\,(=\text{couple at pulley}) \qquad = N\left\{1 - \cdot 92544 \cos n_0 l \frac{Jt}{l} - \cdot 04416 \cos n_1 l \frac{Jt}{l}\right\},$$

$$\Omega\,(=\text{spin of pulley}) \qquad = \omega - \frac{2JN}{K}\left\{\cdot 23288 \sin n_0 l \frac{Jt}{l} + \cdot 00162 \sin n_1 l \frac{Jt}{l}\right\},$$

and for Case (β)

$$M_l = N\left\{1 - 1\cdot 001461 \cos n_0 l \frac{Jt}{l} + \cdot 001773 \cos n_1 l \frac{Jt}{l}\right\},$$

$$M_0 = N\left\{1 - \cdot 997090 \cos n_0 l \frac{Jt}{l} - \cdot 001773 \cos n_1 l \frac{Jt}{l}\right\},$$

$$\Omega = \omega - \frac{2JN}{K}\left\{\cdot 046756 \sin n_0 l \frac{Jt}{l} + \cdot 000025 \sin n_1 l \frac{Jt}{l}\right\},$$

where $n_0 l$ and $n_1 l$ have the values given in the table above. The values of M_l/N, M_0/N and the variation in Ω, *i.e.* $\Omega - \omega$, are represented graphically in Plate XI, Fig. XVI., Cases (α) and (β). In revolutions per minute we have $2JN/K = 1138\cdot 32$ (assuming $J = 10000$ ft. per sec.) and so in

Case (α) $\Omega = 100 - 265\cdot 09 \sin n_0 l\, Jt/l - 1\cdot 84 \sin n_1 l\, Jt/l,$

Case (β) $\Omega = 100 - 53\cdot 22 \sin n_0 l\, Jt/l - \cdot 028 \sin n_1 l\, Jt/l.$

Thus we see a fundamental difference between Cases (α) and (β). In the former the spin varies from 367 to -167 revolutions a minute, or the pulley can take a *negative* spin. In the latter the spin varies from about 153·25 to 46·75 revolutions a minute, but is never negative. In both cases we see that very great irregularity is produced in the rate of the pulley by throwing it into gear with some machine.

It is difficult in Case (α) on the scale of our diagram to allow for the second harmonics. In Case (β) they become quite insensible and the curves for M_l and M_0 fall together. We may safely say that for $\gamma =$ or $< \cdot 01$ the values of M_l and M_0 are practically identical, and their common value may be expressed by the first harmonic

$$M = N\left(1 - c_1 \cos n_0 l \frac{Jt}{l}\right).$$

41. Now suppose the shaft were to be considered as under statical torsion at each instant of time, we should have for the motion of the pulley, rotation ϕ:

$$m\beta^2 \frac{d^2\phi}{dt^2} = -N + K\frac{d\psi}{dx},$$

where $\frac{d\psi}{dx}$ being uniform throughout the shaft would $= -\frac{\phi - \omega t}{l}$.

Hence we deduce

$$\phi = \omega t - \frac{lN}{K} + A \sin\left\{\sqrt{\frac{K}{lm\beta^2}}\,t + C\right\}.$$

Now $$\gamma = \frac{ak^2 l\rho}{m\beta^2} \text{ and } J^2 = \frac{K}{\rho a k^2} = \frac{Kl}{\gamma m\beta^2},$$

if a be here the area of the shaft.

Or $$\phi = \omega t - \frac{lN}{K} + A \sin\left\{\sqrt{\gamma}\frac{Jt}{l} + C\right\}.$$

But $$\phi = 0 \text{ and } \frac{d\phi}{dt} = \omega, \text{ when } t = 0^*,$$

$$\therefore\ \phi = \omega t - \frac{lN}{K}\left\{1 - \cos\sqrt{\gamma}\frac{Jt}{l}\right\}.$$

This gives us

$$M = -K\frac{\phi - \omega t}{l} = N\left\{1 - \cos\sqrt{\gamma}\frac{Jt}{l}\right\},$$

for all points of shaft.

Thus we should be taking

$c_1 = 1$ instead of 1·00167 (Table V),
$c_2 = 1$ instead of ·99667 ,,
$n_0 l = \sqrt{\gamma} = \cdot 1$ instead of ·09983 ,,

in the case of $\gamma = \cdot 01$; that is to say, these differences are practically imperceptible. Or, we may argue that for $\gamma =$ or $< \cdot 01$ we may treat the shaft as taking at each instant the statical torsion due to the couple applied and that we shall thus obtain a close approximation to the first harmonic term in the stress-couple.

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* Physically of course this state of things could not be set up until $t = l/J$, the time the wave takes to traverse shaft. This is allowed for in the form given on p. 29.

Further we have for the spin

$$\frac{d\phi}{dt} = \omega - \frac{NJ}{K}\sqrt{\gamma}\sin\sqrt{\gamma}\frac{Jt}{l}$$

$$= \omega - \frac{2NJ}{K} \cdot \frac{\sqrt{\gamma}}{2}\sin\sqrt{\gamma}\frac{Jt}{l}.$$

This replaces $c_3 = {\cdot}04992$ for $\gamma = {\cdot}01$ by $\sqrt{\gamma}/2$ or $\cdot05$, so that this gives us also a very close expression for the variation in the spin.

Plate XI., Fig. XVI. Case (a) for $\gamma = {\cdot}2346$ shews how the curves for M_0 and M_l begin to separate and at the same time negative values occur for M_l as the second and third harmonics have a more measurable effect.

These results are still more clearly evidenced in Plate VII., Fig. XI. of our Art. 22, which is plotted for $\gamma = {\cdot}5$, where we also remark how much more frequently the maxima begin to follow each other.

42. In Plate XI., Fig. XV. I have plotted so far as the scale would admit the values of M_l and M_0, and $\Omega - \omega$ for $\gamma = 1$. We see here a very considerable divergence between the stress-couples at the pulley (M_0) and at the rotated end (M_l) of the shaft. The latter has a very much greater positive maximum,—its limiting value being $= N(1 + \Sigma c_1) = 2{\cdot}37N$,—and it approaches fairly closely to this limit three times before $Jt/l = 19$. Thus for a value of Jt/l between 18 and 19 we have $M_l = 2{\cdot}312N$. The positive maximum limit of M_0 on the other hand is not closely approached till Jt/l is much greater than the 34 of the diagram, not indeed until Jt/l equals about 191, or indeed a better maximum could be reached at the double of this. In the shaft 50 feet long dealt with in the previous example this is equivalent to saying that the maximum will not be reached until $191 \times 50/10000$ (or even $382 \times 50/10000$) secs. after the pulley has been thrown into gear; this amounts to nearly 1 second (or 2 seconds), within which time viscosity has had some influence in reducing the maximum. Thus the real *maximum-maximorum* may lie much nearer the beginning of the oscillations.

We see further that the negative maximum of M_l is not nearly reached before $Jt/l = 34$. M_l does not indeed theoretically approach its negative limit before $Jt/l =$ about 197, so that again viscosity will have generally had considerable influence. M_0 on the other hand never becomes negative and approaches fairly closely a zero value between $Jt/l = 14$ and 15, and $Jt/l = 22$ and 23. The variation in $\Omega - \omega$ as here plotted shews only a very slight divergence from the first harmonic term. Indeed for all practical purposes for $\gamma = 0 < 1$ we may represent the variation in the spin by the first harmonic. The couples on the other hand requiring two, three or even four harmonics to plot them even to the above scale.

M_l should be equal to zero from $Jt/l = 0$ to 1, its slight divergence from this value in the drawing is owing to the difficulty of considering a sufficient number of harmonics on the scale of the figure.

We may conclude then that for practical purposes a safe rule would be obtained from the following formulae, for $\gamma =$ or < 1:

Maximum stress-couple at pulley $M_0 = N\{1 + \Sigma c_2\}$,

Maximum stress-couple at rotated end $M_l = N\{1 + \Sigma c_1\}$ as positive value,

$= N\{1 - \Sigma c_1\}$ as negative value.

Maximum variation in spin at pulley $\Omega - \omega = \pm \dfrac{2JN}{K} \cdot c_3$ radians per second

$= \dfrac{60}{\pi} \dfrac{JN}{K} \cdot c_3$ revolutions per minute,

where J = velocity of sound in shaft = about 10000 feet per second for a wrought-iron shaft,

N = frictional resistance-couple at pulley,

K = slide modulus × cross-sectional moment of inertia about axis of shaft, when the cross-section is circular*,

and the values of c_1, c_2, c_3 are tabulated for various values of γ in Tables III, IV, V, above.

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* Otherwise the value of K = slide modulus × ν, where ν is given very approximately by Equation (iv) of Art. 3.

Description of Lithographed Tables and Plates.

Table I (*a*). Values of $\phi(\zeta)$ and $\phi'(\zeta)$, Art. 20. Giving effect of throwing shaft-pulley into gear.

Table I (*b*). Values of $\phi(\zeta)$ and $\phi'(\zeta)$, Art. 28. Giving effect of suddenly applying brakes to wheels on an axle.

Plate I. Fig. I. Impulsive couple on shaft. Figs. II. and III. Impulsive couple of finite duration.

Plate II. Fig. IV. Effect of Repeated loading and unloading, two loads. Figs. VI. and VII. Effect of progressive but rapid application of load, Art. 33.

Plate III. Fig. V. Effect of four and six repeated loadings and unloadings on a shaft.

Plate IV. Fig. VIII. Effect of impulsive torsional couples on shafts with fixed terminals.

Plate V.
Plate VI.
Plate VII.
Effect on shafts when pulleys are thrown suddenly into gear; rotated angles, spins and stresses. Figs. IX to XI.

Plate VIII.
Plate IX.
Plate X.
Effect on axles, when wheels are suddenly braked; rotated angles, spins and stresses. Figs. XII to XIV.

Plate XI. Effect of suddenly throwing a shaft-pulley into gear; variations in spin and maximum stresses determined by normal functions. Figs. XV and XVI.

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Table I. Stress in Shafting thrown suddenly into gear. Values of $\varphi(\xi)$ and $\varphi'(\xi)$ from $\xi = 0$ to $\xi = 10l$.

$$f(\xi) = \frac{N \cdot l}{K \cdot \gamma}\, \varphi(\xi) . \qquad f'(\xi) = \frac{N}{K}\, \varphi'(\xi) .$$

ξ/l	$\gamma = 1$		$\gamma = 1/2$		$\gamma = 1/10$		$\gamma = 1/100$		$\gamma = 1/1000$	
	$\varphi(\xi)$	$\varphi'(\xi)$	$\varphi(\xi)$	$\varphi'(\xi)$	$\varphi(\xi)$	$\varphi'(\xi)$	$\varphi(\xi)$	$\varphi'(\xi)$	$\varphi(\xi)$	$\varphi'(\xi)$
0	0	0	0	0	0	0	0	0	0	0
·25	−·028,801	−·221,199	−·007,497	−·117,503	−·000,310	−·024,690	−·000,003	−·002,497	−·000,000	−·000,250
·5	−·106,531	−·393,469	−·028,801	−·221,199	−·001,229	−·048,771	−·000,012	−·004,988	−·000,000	−·000,500
·75	−·222,367	−·527,633	−·062,289	−·312,711	·002,743	−·072,257	−·000,028	−·007,472	−·000,000	−·000,750
1	−·367,879	−·632,121	−·106,531	−·393,469	−·004,837	−·095,163	−·000,050	−·009,950	−·000,000	−·001,000
1·25	−·536,505	−·713,495	−·160,261	−·464,739	−·007,497	−·117,503	−·000,078	−·012,422	−·000,001	−·001,249
1·5	−·723,130	−·776,870	−·222,367	−·527,633	−·010,708	−·139,292	−·000,112	−·014,888	−·000,001	−·001,499
1·75	−·923,774	−·826,226	−·291,862	−·583,138	−·014,457	−·160,543	−·000,152	−·017,348	−·000,002	−·001,748
2	−1·135,335	−·864,665	−·367,879	−·632,121	−·018,731	−·181,269	−·000,199	−·019,801	−·000,002	−·001,998
2·25	−1·379,595	−1·062,803	−·456,537	−·778,469	−·023,820	−·225,560	−·000,254	−·024,740	−·000,003	−·002,497
2·5	−1·655,961	−1·130,977	−·560,701	−·881,697	−·029,991	−·267,557	−·000,324	−·029,653	−·000,003	−·002,997
2·75	−1·938·276	−1·116,990	−·675,506	−·949,915	−·037,181	−·307,333	−·000,403	−·034,541	−·000,004	−·003,495
3	−2·210,392	−1·053,850	−·797,006	−·989,932	−·045,340	−·344,986	−·000,494	−·039,406	−·000,004	−·003,994
3·25	−2·462,996	−·963,994	−·922,053	−1·007,425	−·054,412	−·380,594	−·000,598	−·044,246	−·000,005	−·004,493
3·5	−2·691,417	−·862,323	−1·048,122	−1·007,144	−·064,352	−·414,232	−·000,716	−·049,060	−·000,006	−·004,989
3·75	−2·893,987	−·758,465	−1·173,260	−·993,016	−·075,108	−·445,978	−·000,845	−·053,857	−·000,007	−·005,487
4	−3·070,971	−·658,359	−1·295,940	−·968,302	−·086,635	−·475,903	−·000,984	−·058,618	−·000,009	−·005,986
4·25	−3·243,928	−·689,280	−1·421,321	−1·025,611	−·099,191	−·527,549	−·001,141	−·065,845	−·000,010	−·006,735
4·5	−3·403,412	−·571,605	−1·549,779	−1·021,218	−·112,972	−·574,589	−·001,313	−·073,017	−·000,013	−·007,481
4·75	−3·525,614	−·403,100	−1·674,860	−·974,547	−·127,880	−·617,272	−·001,505	−·080,133	−·000,014	−·008,225
5	−3,603·,324	−·238,134	−1·792,288	−·900,639	−·143,801	−·655,844	−·001,717	−·087,195	−·000,017	−·008,972
5·25	−3·647,274	−·103,725	−1·899,370	−·811,002	−·160,641	−·690,546	−·001,947	−·094,201	−·000,020	−·009,716
5·5	−3·661,632	−·010,274	−1·994,747	−·714,275	−·178,300	−·721,580	−·002,187	−·101,157	−·000,022	−·010,462
5·75	−3·655,898	+·041,420	−2·077,919	−·616,898	−·196,713	−·749,177	−·002,449	−·108,058	−·000,026	−·011,204
6	−3·643,038	+·055,650	−2·149,111	−·523,051	−·215,734	−·773,535	−·002,731	−·114,904	−·000,028	−·011,950
6·25	−3 647,262	−·048,091	−2,214,791	−·514,500	−·235,633	−·817,724	−·003,026	−·124,179	−·000,031	−·012,941
6·5	−3·655,870	−·012,325	−2·275,276	−·446,165	−·256,567	−·855,061	−·003,350	−·133,354	−·000,034	−·013,933
6·75	−3·653,439	+·024,953	−2·325,184	−·349,502	−·278,340	−·886,053	−·003,694	−·142,434	−·000,039	−·014,922
7	−3·647,173	+·074,363	−2·363,369	−·245,983	−·300,818	−·911,225	−·004,059	−·157,420	−·000,040	−·015,912
7·25	−3·657,148	−·046,969	−2·386,982	−·228,845	−·323,855	−·931,043	−·004,446	−·160,313	−·000,045	−·016,902
7·5	−3·678,114	−·170,988	−2,400,443	−·069,085	−·347,327	−·945,952	−·004,860	−·169,110	−·000,050	−·017,889
7·75	−3·738,910	−·313,772	−2·405,326	−·008,826	−·371,110	−·956,372	−·005,295	−·177,816	−·000,054	−·018,875
8	−3·836,280	−·465,005	−2·403,578	+·029,438	−·395,109	−·962,293	−·005,745	−·186,430	−·000,060	−·019,860
8·25	−3·984,389	−·666,441	−2·403,918	−·020,793	−·419,575	−·987,575	−·006,227	−·197,424	−·000,067	−·021,094
8·5	−4·156,237	−·700,220	−2·406,446	−·006,111	−·444,410	−1·004,073	−·006,735	−·208,270	−·000,070	−·022,327
8·75	−4·341,740	−·735,620	−2·406,867	+·007,547	−·469,650	−1·013,065	−·007,267	−·218,972	−·000,075	−·023,559
9	−4·527,508	−·810,957	−2·404,392	+·021,316	−·495,025	−1·015,391	−·007,831	−·229,530	−·000,083	−·024,787
9·25	−4·742·576	−·911,580	−2·402,415	+·014,745	−·520,367	−1·011,822	−·008,417	−·239,942	−·000,090	−·026,015
9·5	−4·983,094	−1·009,240	−2·402,275	−·016,369	−·545,562	−1·003,062	−·009,031	−·250,217	−·000,095	−·027,242
9·75	−5·244,927	−1·079,938	−2·407,328	−·071,235	−·570,488	−·989,771	−·009,669	−·260,353	−·000,103	−·028,467
10	−5·579,494	−1·109,258	−2·420,901	−·146,057	−·595,023	−·972,559	−·010,337	−·270,345	−·000,109	−·029,691

Table II. stress in axles, when Break is suddenly applied. Values of $\varphi(\xi)$ & $\varphi'(\xi)$ from $\xi=0$

$$f(\xi)=\frac{Nl}{K\gamma}\varphi(\xi) \qquad f(\xi)=\frac{N}{K}\varphi(\xi)$$

ξ/l	$\gamma=1$		$\gamma=1/2$		$\gamma=1/10$		$\gamma=1/100$		$\gamma=1/1000$	
	$\varphi(\xi)$	$\varphi'(\xi)$	$\varphi(\xi)$	$\varphi'(\xi)$	$\varphi(\xi)$	$\varphi'(\xi)$	$\varphi(\xi)$	$\varphi'(\xi)$	$\varphi(\xi)$	$\varphi'(\xi)$
−1 to +1	0	0	0	0	0	0	0	0	0	0
1.25	−.028,801	−.221,199	−.007,497	−.117,505	−.000,310	−.024,690	−.000,003	−.002,497	−.000,000	−.000,250
1.5	−.106,531	−.393,469	−.028,801	−.221,199	−.001,229	−.048,771	−.000,012	−.004,988	−.000,000	−.000,500
1.75	−.222,367	−.527,633	−.062,289	−.312,711	−.002,743	−.072,257	−.000,028	−.007,472	−.000,000	−.000,750
2	−.367,879	−.632,121	−.106,531	−.393,469	−.004,837	−.095,163	−.000,060	−.009,950	−.000,000	−.001,000
2.25	−.536,505	−.713,495	−.160,261	−.464,739	−.007,497	−.117,503	−.000,078	−.012,422	−.000,001	−.001,249
2.5	−.723,130	−.776,870	−.222,367	−.527,633	−.010,708	−.139,292	−.000,112	−.014,888	−.000,001	−.001,499
2.75	−.923,774	−.826,226	−.291,862	−.583,138	−.014,457	−.160,543	−.000,152	−.017,348	−.000,002	−.001,748
3	−1.135,335	−.864,665	−.367,879	−.632,121	−.018,731	−.181,269	−.000,199	−.019,801	−.000,002	−.001,998
3.25	−1.331,203	−.726,399	−.442,767	−.572,227	−.023,212	−.177,408	−.000,248	−.019,758	−.000,003	−.001,997
3.5	−1.508,209	−.704,853	−.512,309	−.545,293	−.027,611	−.174,847	−.000,296	−.019,727	−.000,003	−.001,997
3.75	−1.689,580	−.755,154	−.580,174	−.544,405	−.031,963	−.173,523	−.000,347	−.019,709	−.000,004	−.001,997
4	−1.889,182	−.846,576	−.649,254	−.563,808	−.036,296	−.173,378	−.000,396	−.019,704	−.000,004	−.001,998
4.25	−2.114,552	−.958,458	−.721,771	−.598,751	−.040,642	−.174,352	−.000,444	−.019,712	−.000,005	−.001,997
4.5	−2.368,977	−1.077,283	−.799,426	−.645,308	−.045,024	−.176,392	−.000,494	−.019,730	−.000,006	−.001,999
4.75	−2.653,049	−1.194,499	−.883,450	−.700,274	−.049,470	−.179,444	−.000,543	−.019,761	−.000,007	−.001,999
5	−2.965,661	−1.305,009	−.974,730	−.761,028	−.054,005	−.183,457	−.000,594	−.019,804	−.000,007	−.001,998
5.25	−3.324,914	−1.529,890	−1.080,167	−.915,373	−.058,951	−.211,853	−.000,645	−.022,345	−.000,008	−.002,247
5.5	−3.716,972	−1.586,585	−1.201,333	−1.015,682	−.064,580	−.238,185	−.000,705	−.024,865	−.000,008	−.002,499
5.75	−4.112,390	−1.572,036	−1.332,520	−1.078,247	−.070,842	−.262,598	−.000,771	−.027,367	−.000,008	−.002,751
6	−4.501,490	−1.541,116	−1.469,668	−1.115,599	−.077,691	−.285,226	−.000,841	−.029,851	−.000,009	−.003,000
6.25	−4.883,980	−1.522,115	−1.610,770	−1.137,250	−.085,091	−.306,200	−.000,915	−.032,317	−.000,010	−.003,248
6.5	−5.265,572	−1.527,122	−1.753,809	−1.150,309	−.092,992	−.325,646	−.001,001	−.034,763	−.000,011	−.003,498
6.75	−5.649,776	−1.558,774	−1.898,211	−1.159,906	−.101,381	−.343,665	−.001,093	−.037,190	−.000,012	−.003,746
7	−6.045,990	−1.614,662	−2.043,801	−1.169,901	−.110,166	−.360,383	−.001,191	−.039,600	−.000,012	−.003,994
7.25	−6.442,104	−1.602,921	−2.185,023	−1.104,772	−.119,077	−.353,022	−.001,286	−.039,515	−.000,013	−.003,995
7.5	−6.861,812	−1.768,375	−2.323,032	−1.112,621	−.127,839	−.348,545	−.001,386	−.039,454	−.000,014	−.003,995
7.75	−7.328,421	−1.960,933	−2.464,994	−1.164,280	−.136,524	−.347,721	−.001,486	−.039,418	−.000,015	−.003,994
8	−7.839,741	−2.120,633	−2.622,055	−1.239,193	−.145,192	−.347,275	−.001,581	−.039,408	−.000,016	−.003,995
8.25	−8.384,336	−2.235,861	−2.775,176	−1.323,345	−.153,905	−.349,985	−.001,678	−.039,423	−.000,017	−.003,994
8.5	−8.950,130	−2.287,860	−2.945,901	−1.407,599	−.162,711	−.354,640	−.001,776	−.039,462	−.000,018	−.003,995
8.75	−9.525,558	−2.315,442	−3.127,074	−1.486,402	−.171,652	−.361,042	−.001,877	−.039,524	−.000,019	−.003,995
9	−10.105,886	−2.325,469	−3.317,124	−1.556,776	−.180,777	−.369,005	−.001,977	−.039,610	−.000,020	−.003,996
9.25	−10.701,405	−2.387,777	−3.520,894	−1.684,357	−.190,403	−.400,651	−.002,079	−.042,194	−.000,021	−.004,248
9.5	−11.288,403	−2.305,034	−3.734,670	−1.734,075	−.200,776	−.428,507	−.002,187	−.044,744	−.000,022	−.004,497
9.75	−11.864,162	−2.259,078	−3.950,601	−1.725,115	−.211,798	−.453,003	−.002,303	−.047,262	−.000,023	−.004,747
10	−12.424,862	−2.294,247	−4.165,912	−1.710,492	−.223,403	−.474,521	−.002,425	−.049,752	−.000,024	−.004,999
10.25	−13.010,088	−2.396,724	−4.378,359	−1.698,471	−.235,507	−.493,416	−.002,553	−.052,208	−.000,026	−.005,247
10.5	−13.626,208	−2.535,522	−4,590,493	−.697,809	−.248,058	−.510,014	−.002,683	−.054,637	−.000,027	−.005,496
10.75	−14.278,545	−2.681,576	−4.803,602	−1.711,959	−.260,992	−.524,614	−.002,825	−.057,031	−.000,020	−.005,745
11	−14.965,970	−2.814,474	−5.019,089	−1.741,287	−.274,271	−.537,499	−.002,973	−.059,397	−.000,031	−.005,993

FIGURE I.

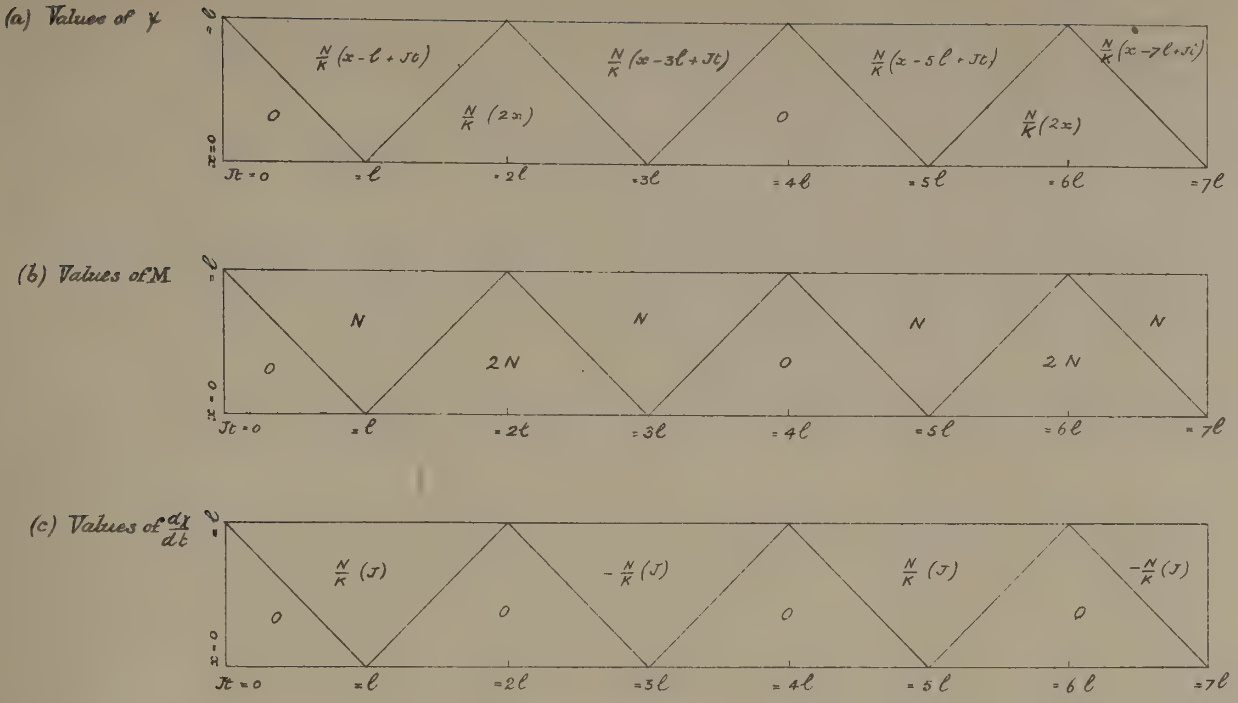


FIGURE II.

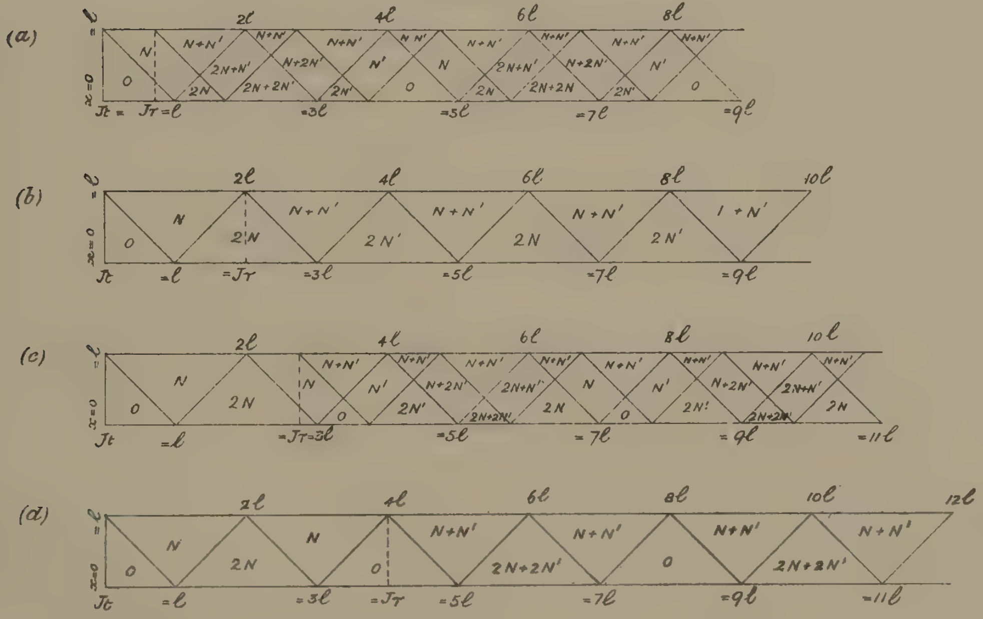


FIGURE III.

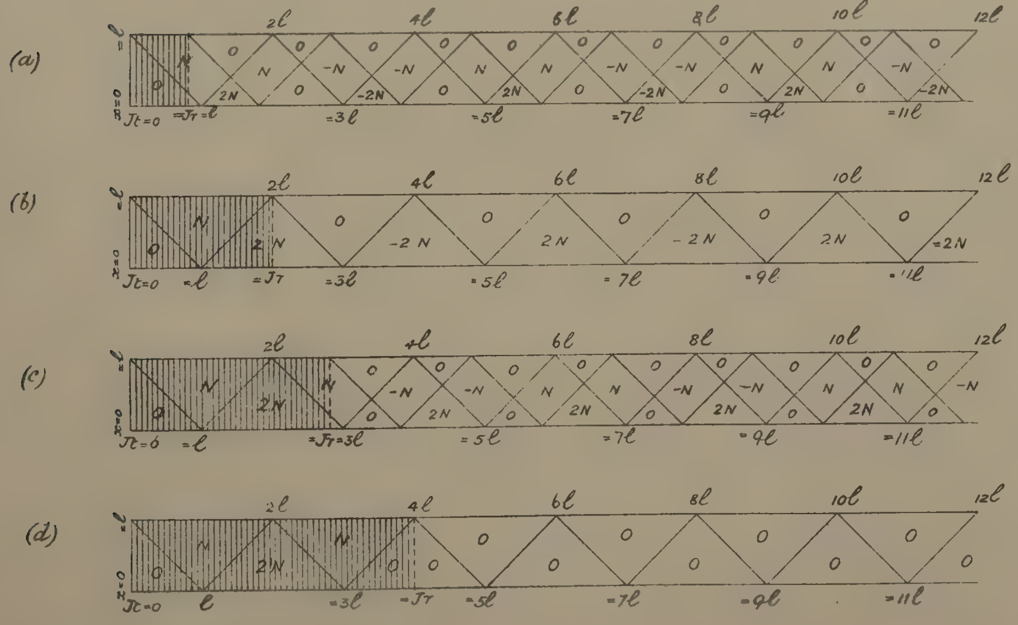


PLATE II

Fig. IV (a)

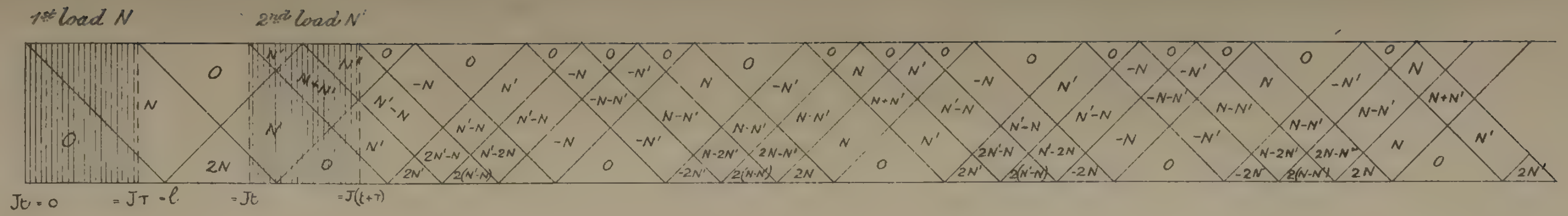
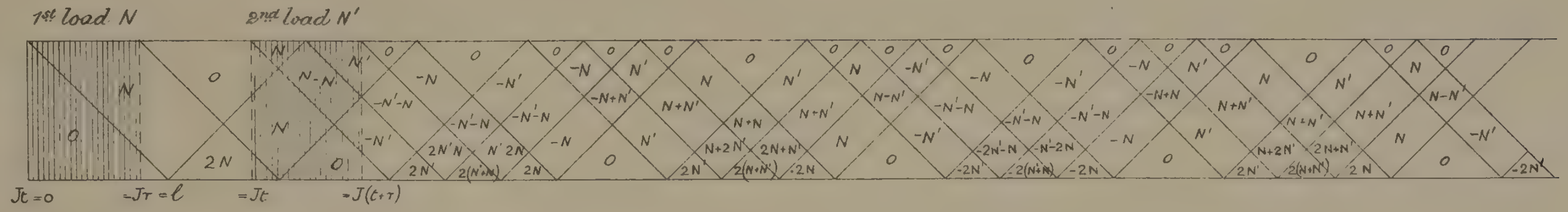
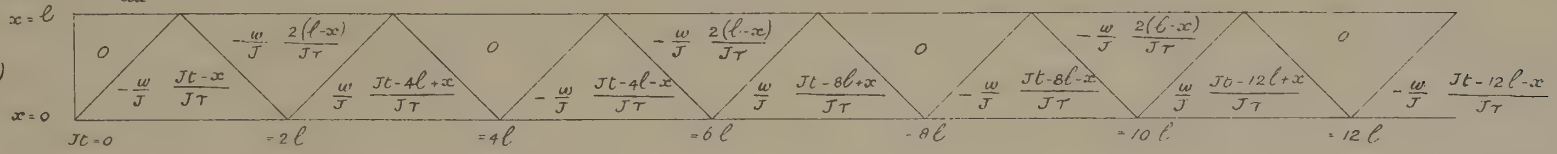


Fig. IV (b)



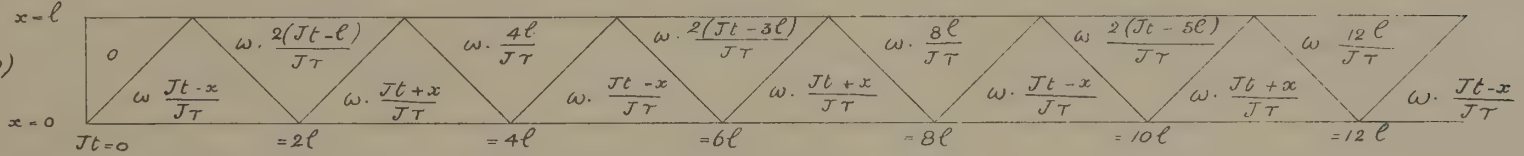
Values of $\frac{d\psi}{dx}$ for $Jt < J\tau$

Fig. VI (a)



$x = l$
$x = 0$
0
$-\frac{\omega}{J}\,\frac{Jt-x}{J\tau}$
$-\frac{\omega}{J}\,\frac{2(l-x)}{J\tau}$
$\frac{\omega}{J}\,\frac{Jt-4l+x}{J\tau}$
$-\frac{\omega}{J}\,\frac{Jt-4l-x}{J\tau}$
$\frac{\omega}{J}\,\frac{Jt-8l+x}{J\tau}$
$-\frac{\omega}{J}\,\frac{Jt-8l-x}{J\tau}$
$\frac{\omega}{J}\,\frac{Jt-12l+x}{J\tau}$
$-\frac{\omega}{J}\,\frac{Jt-12l-x}{J\tau}$
$Jt = 0$
$= 2l$
$= 4l$
$= 6l$
$= 8l$
$= 10l$
$= 12l$

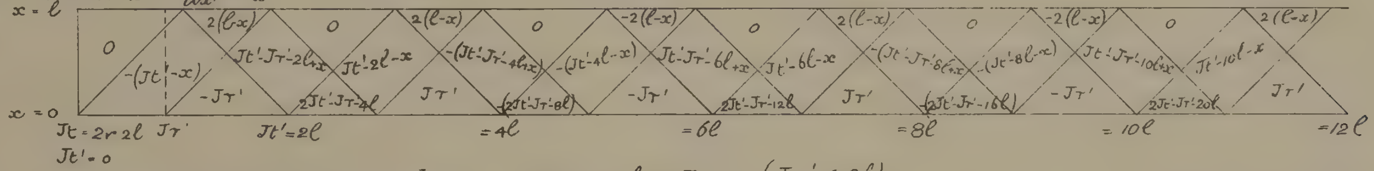
Fig. VI (b)



$x = l$
$x = 0$
0
$\omega\,\frac{Jt-x}{J\tau}$
$\omega\cdot\frac{2(Jt-l)}{J\tau}$
$\omega\cdot\frac{4l}{J\tau}$
$\omega\cdot\frac{Jt+x}{J\tau}$
$\omega\cdot\frac{2(Jt-3l)}{J\tau}$
$\omega\cdot\frac{Jt-x}{J\tau}$
$\omega\cdot\frac{8l}{J\tau}$
$\omega\cdot\frac{2(Jt-5l)}{J\tau}$
$\omega\,\frac{12l}{J\tau}$
$Jt = 0$
$= 2l$
$= 4l$
$= 6l$
$= 8l$
$= 10l$
$= 12l$

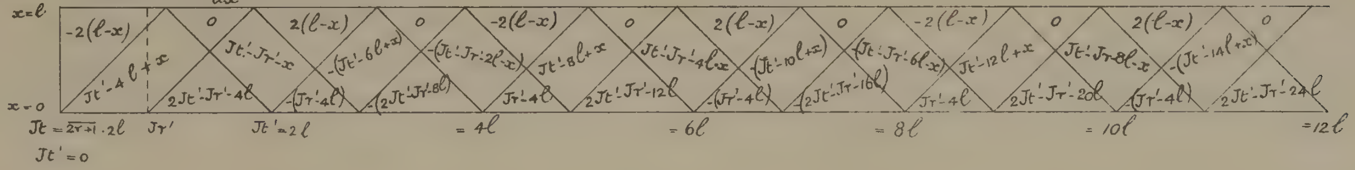
Fig. VIII Values of $\frac{d\psi}{dx} \times \frac{J}{\omega} \times J\tau$ when $J\tau = 2r\,2l + J\tau'$ $(J\tau' < 2l)$

(a)



Values of $\frac{d\psi}{dx} \times \frac{J}{\omega} \times J\tau$, when $J\tau = \overline{2r+1}\;2l + J\tau'$ $(J\tau' < 2l)$

Fig. VII (b)



REPEATED LOADING IN SHAFTING.

Fig V

Fig (a) Stress at built-in end of a shaft for four repeated loadings N, N′ N, N′.

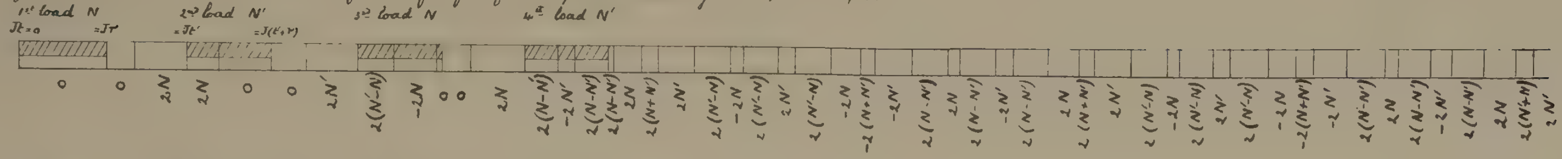


Fig (b) Stress at built-in end of a shaft for four repeated loadings N, -N′, N, -N′.

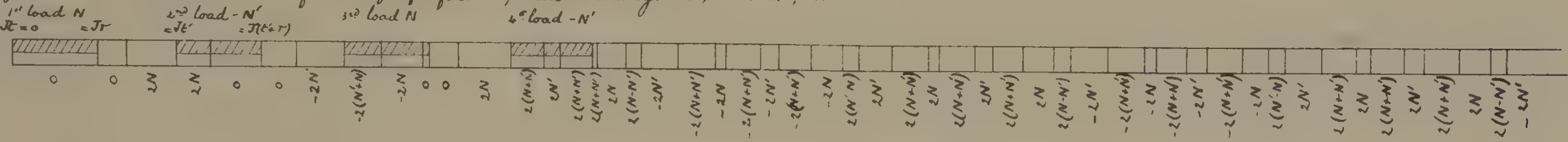


Fig (c) Stress at built-in end of a shaft for six repeated loadings N, N′, N, N′, N, N′.

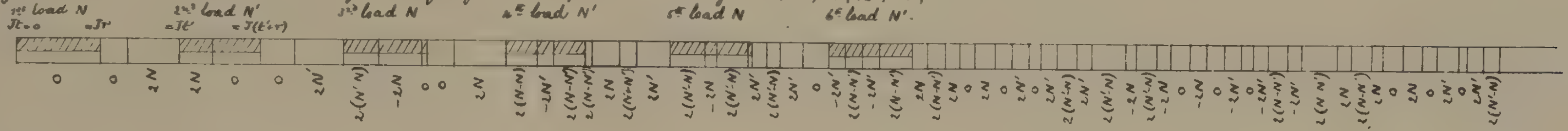
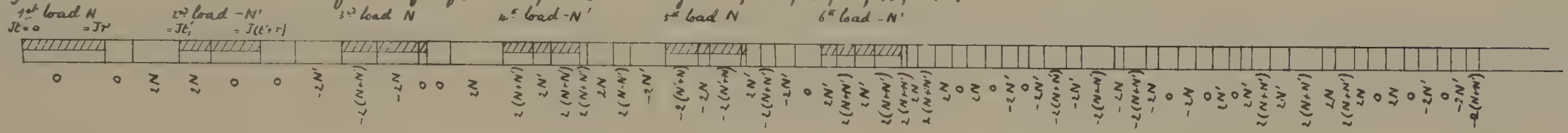


Fig (d). Stress at built-in end of a shaft for six repeated loadings N, -N′, N, -N′, N, -N′.



These figures are only diagrammatic, they give the succession of torsional couples at the built-in end of the shaft, but the duration of these couples or the distances Jt measured along the time-axis have only been roughly plotted out. They are double the scale of previous diagrams. Reduction to ½ scale of origin

PLATE IV

Fig. VIII(a).

ANGLES TURNED THROUGH BY END OF SHAFT STRUCK BEFORE IMPACT CEASES

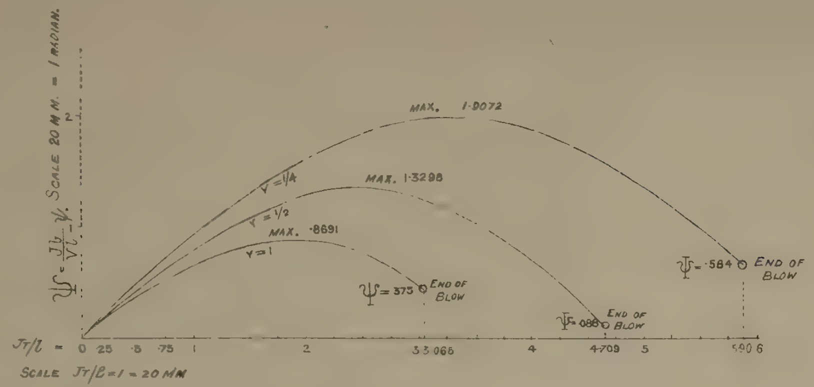


Fig. VIII(b).

TORSION - COUPLES AT BUILT - IN END OF SHAFT

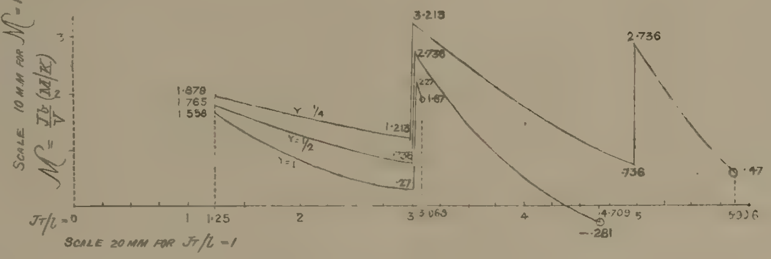
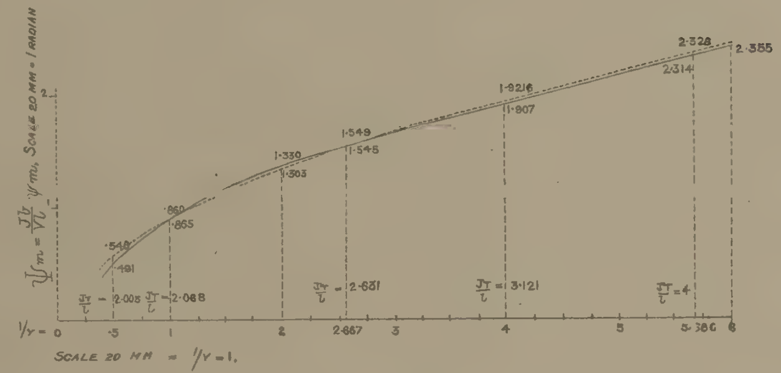


Fig. VIII(c).

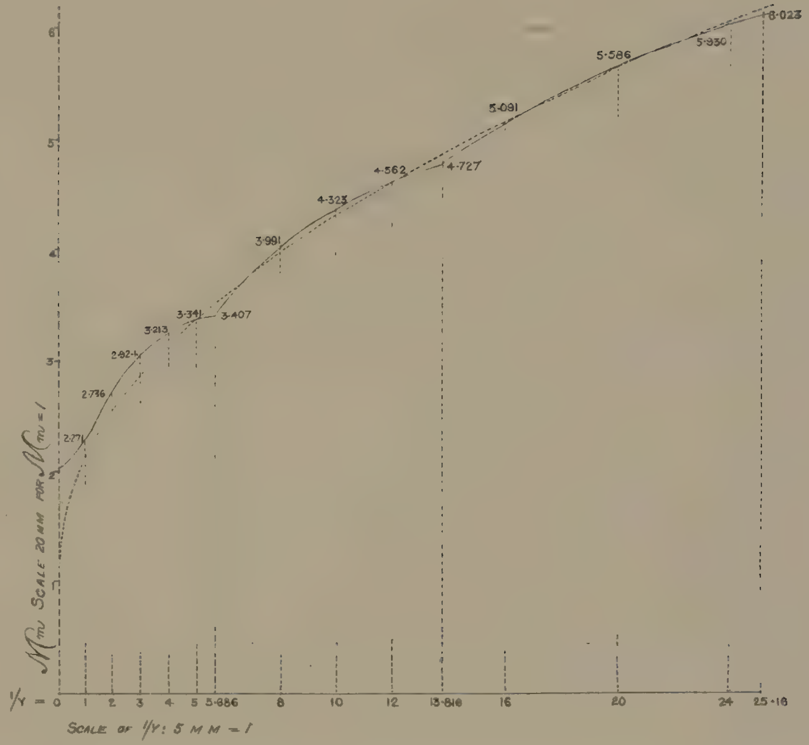
MAXIMUM ANGLES TURNED THROUGH BY END OF SHAFT STRUCK.



THE DOTTED CURVE IS GIVEN BY $\Psi_m = \sqrt{\frac{1/Y}{1+1/3 Y}}$

Fig. VIII(d).

MAXIMUM TORSIONAL COUPLES AT THE BUILT-IN END OF SHAFT



THE DOTTED CURVE IS GIVEN BY $\mathcal{M}_m = \sqrt{1/Y + 1}$

STRESS IN SHAFTING DIAGRAMS OF $\frac{\phi(\zeta)}{\gamma}$ AND $\phi'(\zeta)$

Fig IX(a)

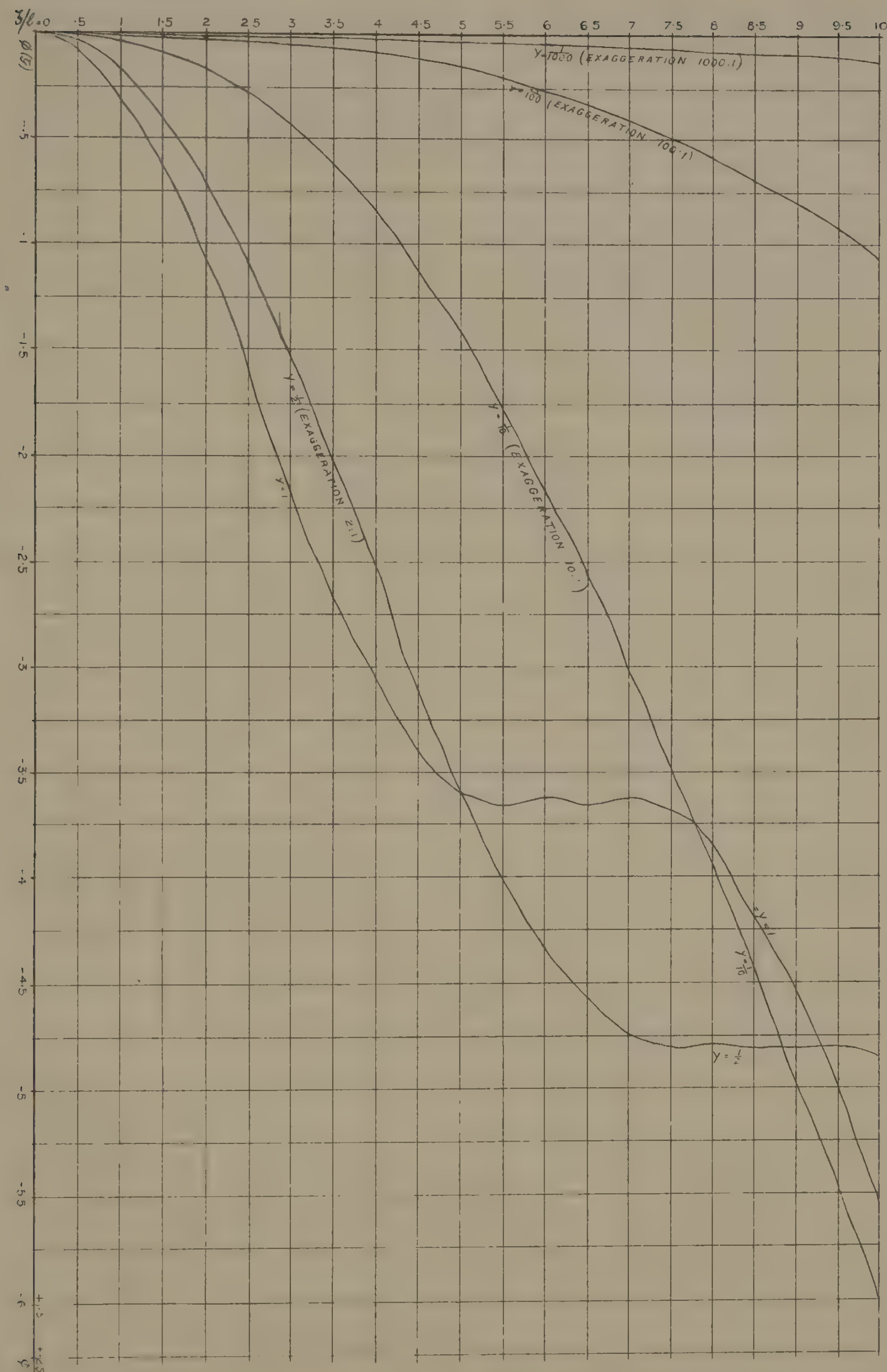
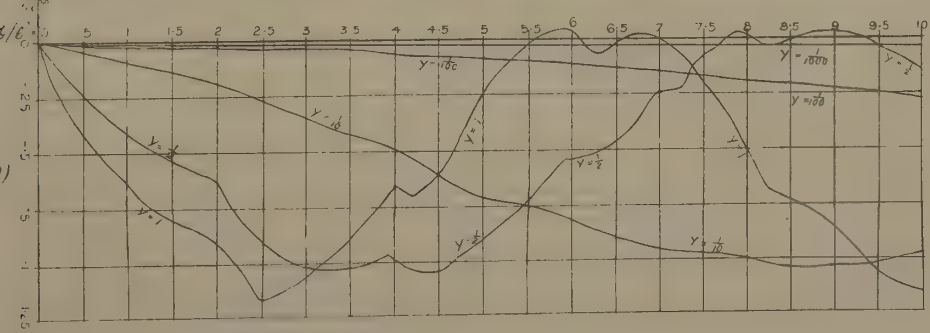


Fig IX(b)



ORIGINAL DRAWING SCALE. $\zeta = 1 = 1{\cdot}8$ CENTIMETRES.

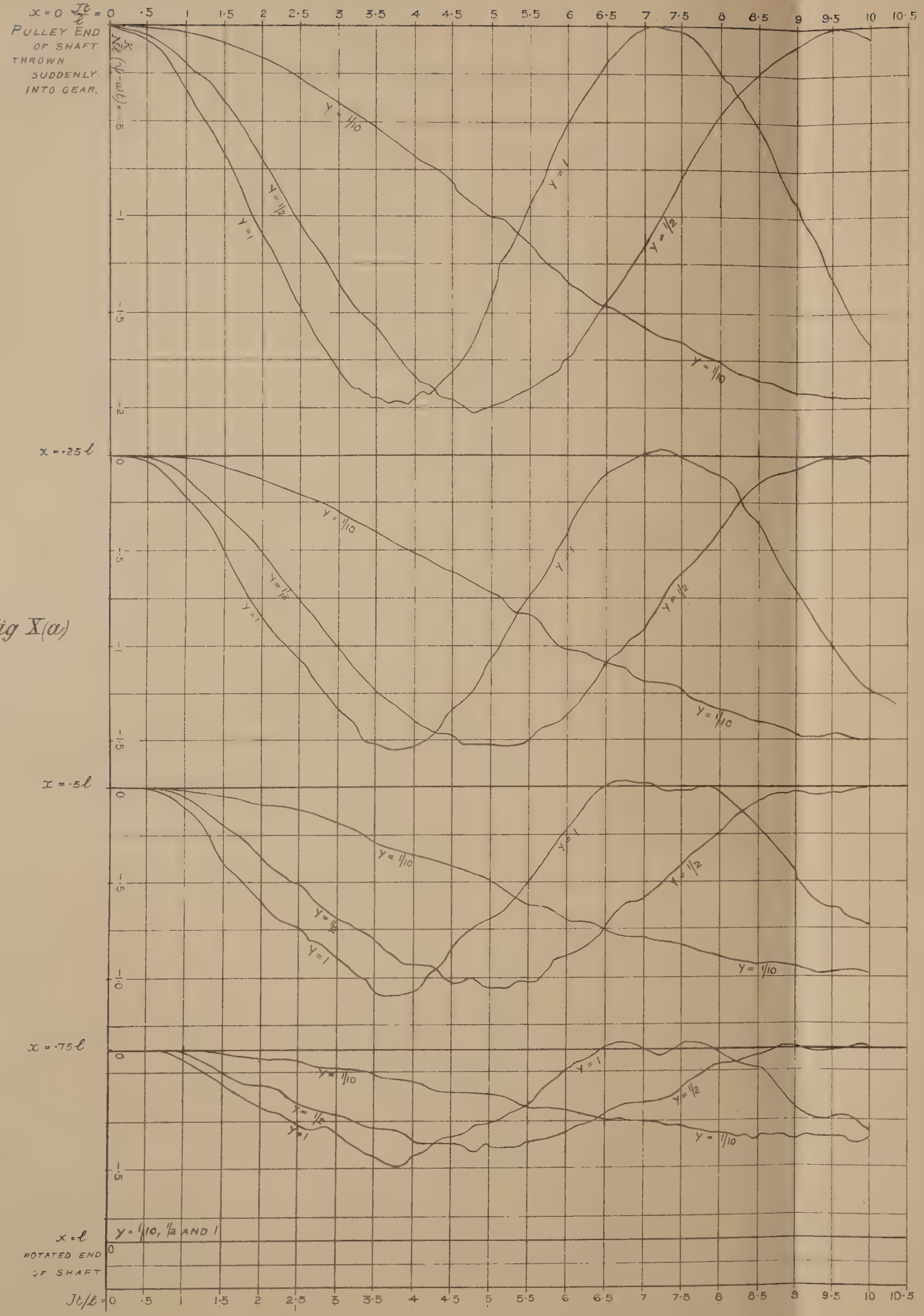
$\frac{\phi(\zeta)}{\gamma} \cdot 1 \cdot 4$ CENTIMETRES

$\phi'(\zeta) \cdot 1 = 4$ CENTIMETRES.

SCALE OF ORIGINAL CENTIMETRES mm 0 2 centimetres

PLATE VI

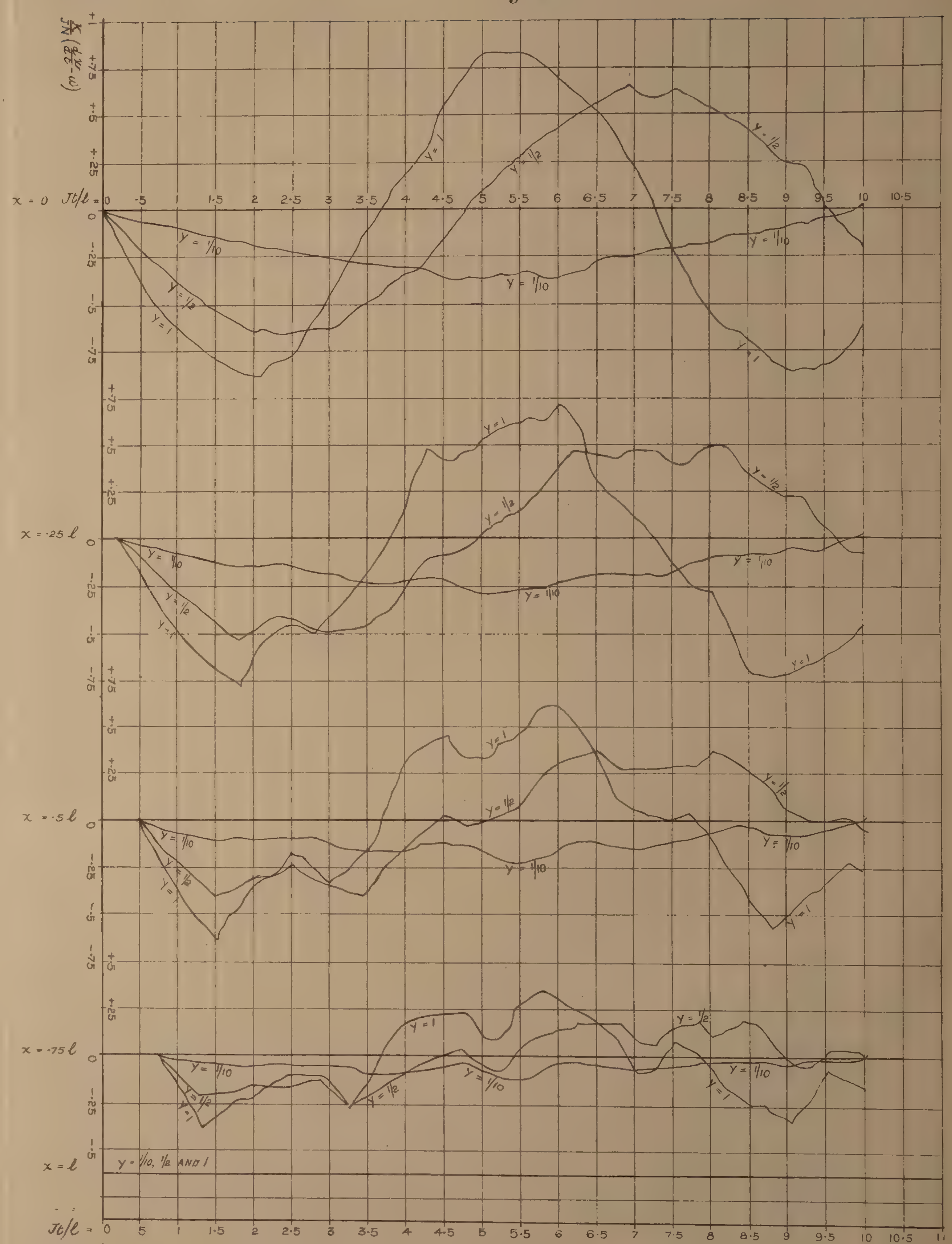
Fig X(a)



SCALE. Unit of Jt/ℓ = 1·6 centimetres. For Angle of Torsion $(\psi-\omega t)$:- $\frac{N\ell}{K}$ radians = 4 cms

STRESS IN SHAFTING. DIAGRAM OF ANGLES OF TORSION AT FIVE POINTS OF THE SHAFTS $y = 1, \frac{1}{2}$ AND $\frac{1}{10}$ FROM $t = 0$ TO $t = 10\ell/J$.

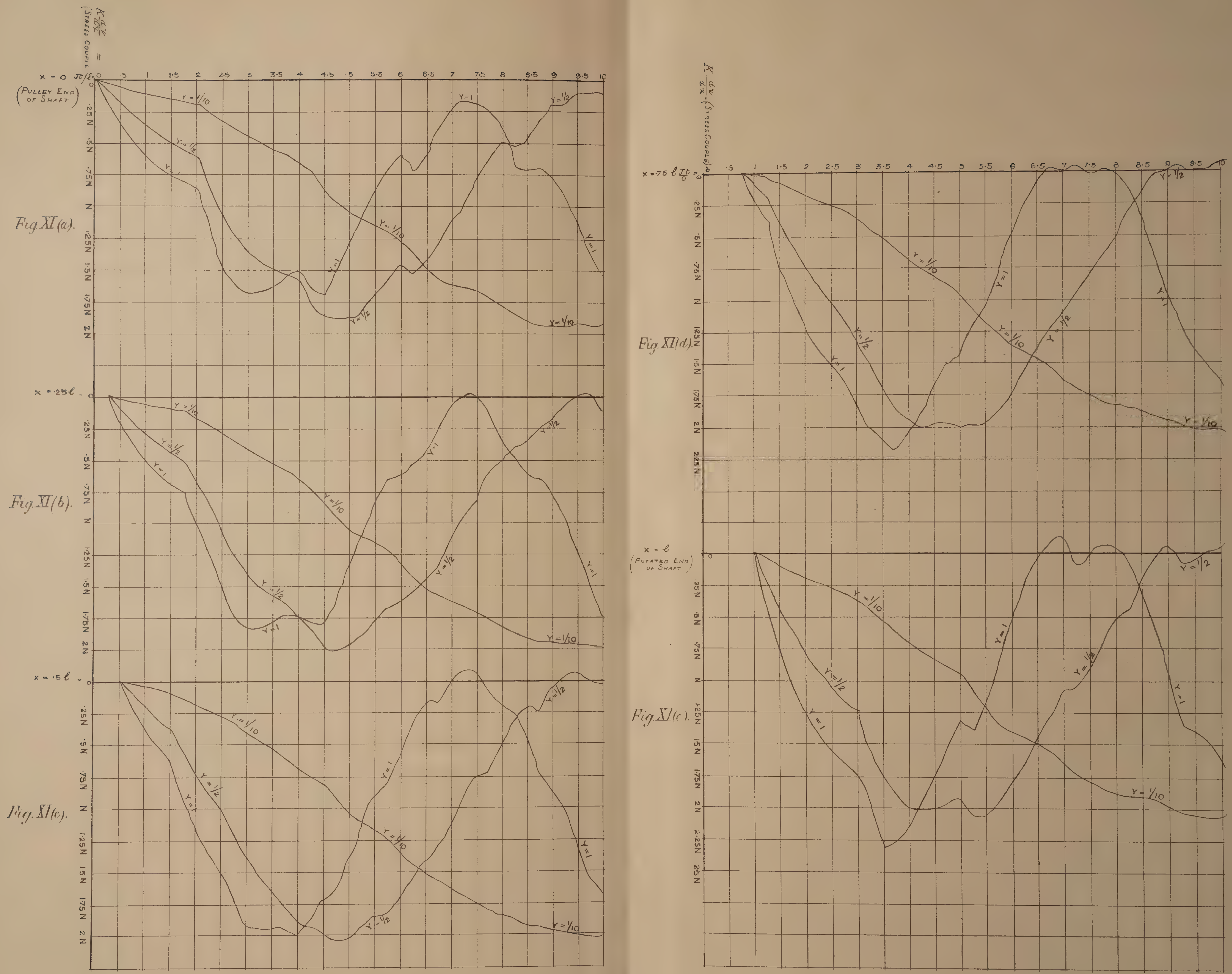
Fig X (b).



SCALE. Unit of Jt/ℓ = 1·6 cms For Variation in spin $\left\{\frac{d\psi}{dt}-\omega\right\}$:- $\frac{JN}{K}$ radians per sec or $\frac{30JN}{\pi K}$ revolutions per min = 4 cms

STRESS IN SHAFTING. DIAGRAM OF VARIATION IN SPIN AT FIVE POINTS OF THE SHAFTS $y = 1, \frac{1}{2}$ AND $\frac{1}{10}$ FROM $t = 0$ TO $t = 10\ell/J$.

SCALE OF ORIGINAL CENTIMETRES 0 1 2 centimetres

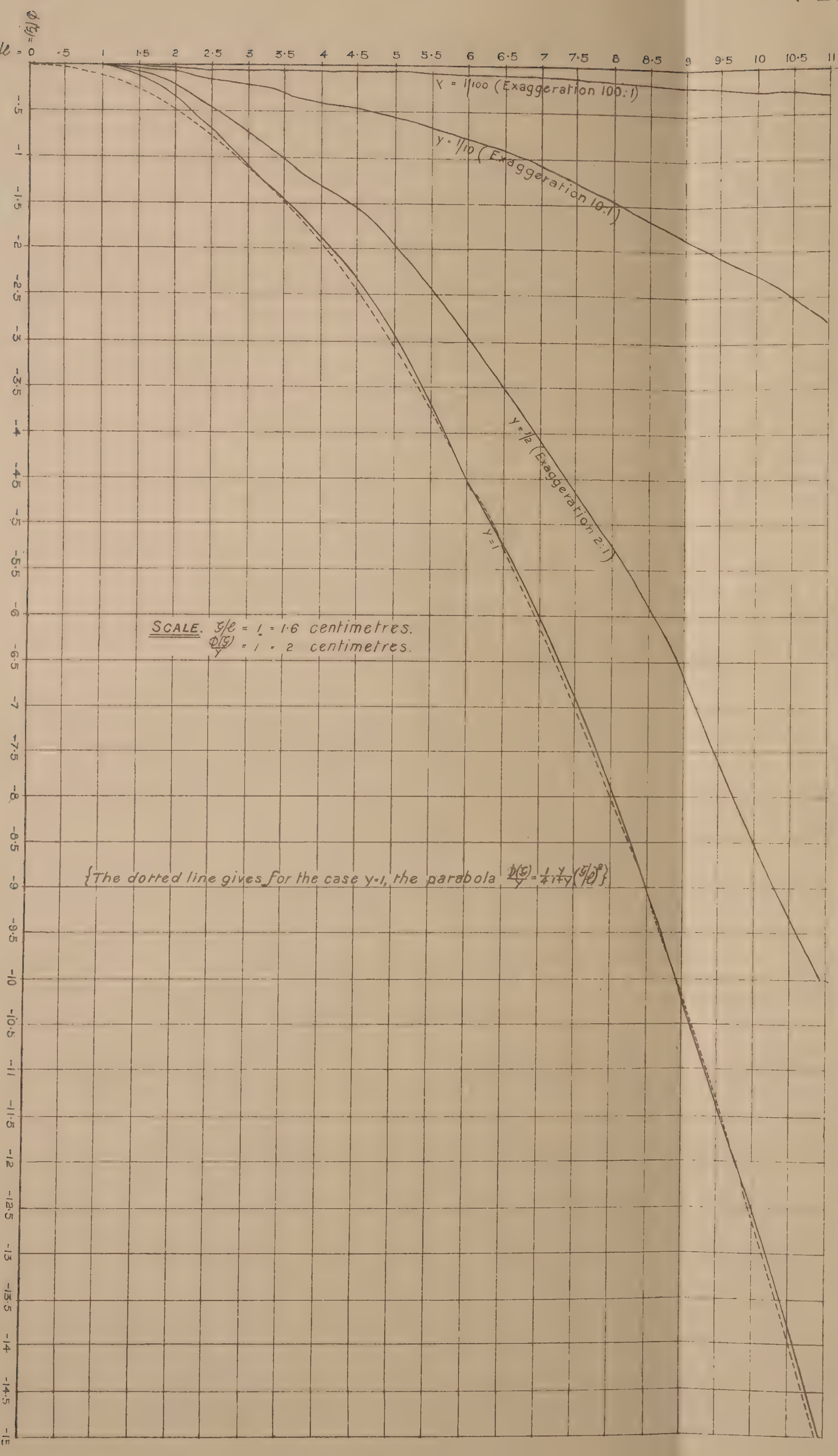


Stress in Shafting, thrown suddenly into gear.
Torsional Couple at five points of a shaft for Y = 1, 1/2 and 1/10 from t = 0, to t = 10ℓ/J.
Scale. Unit of Jt/ℓ = 1·6 centimetres
Torsional couple $K\frac{d\psi}{dx}$ has for unit N = 4 centimetres

Scale of original centimetres

PLATE VIII

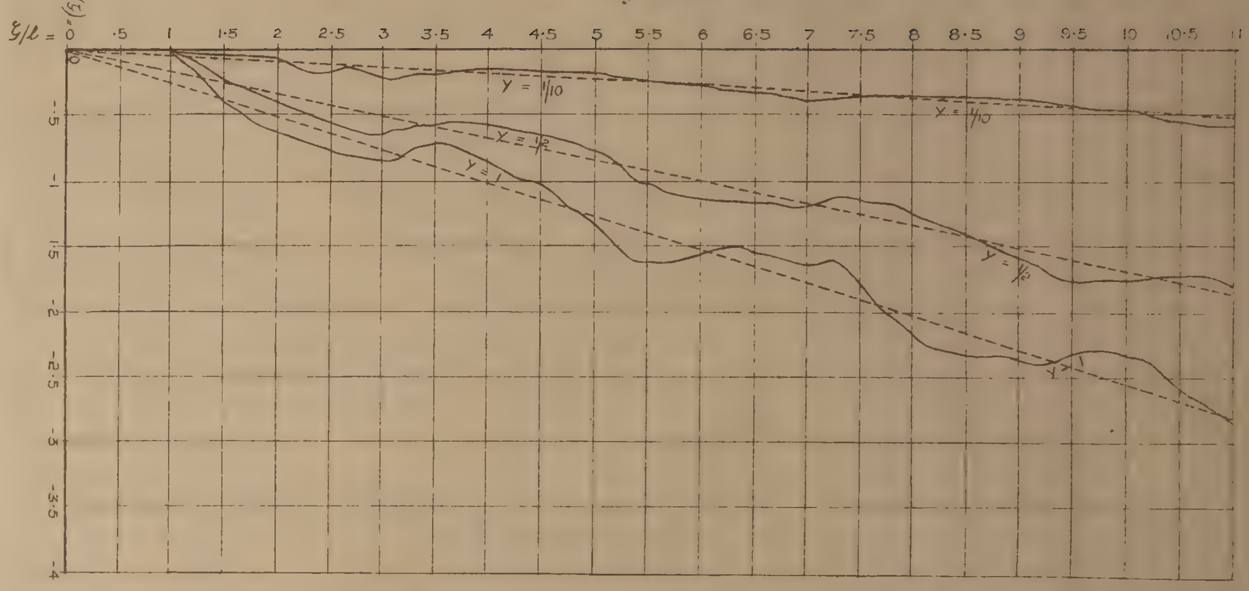
Fig XII(a).



SCALE. $\xi/l = 1 = 1{\cdot}6$ centimetres.
$\frac{\phi(\xi)}{y} = 1 = 2$ centimetres.

{The dotted line gives for the case y = 1, the parabola $\frac{\phi(\xi)}{y} = \frac{1}{4}\frac{y}{1+y}\left(\frac{\xi}{l}\right)^2$}

Fig XII(b).

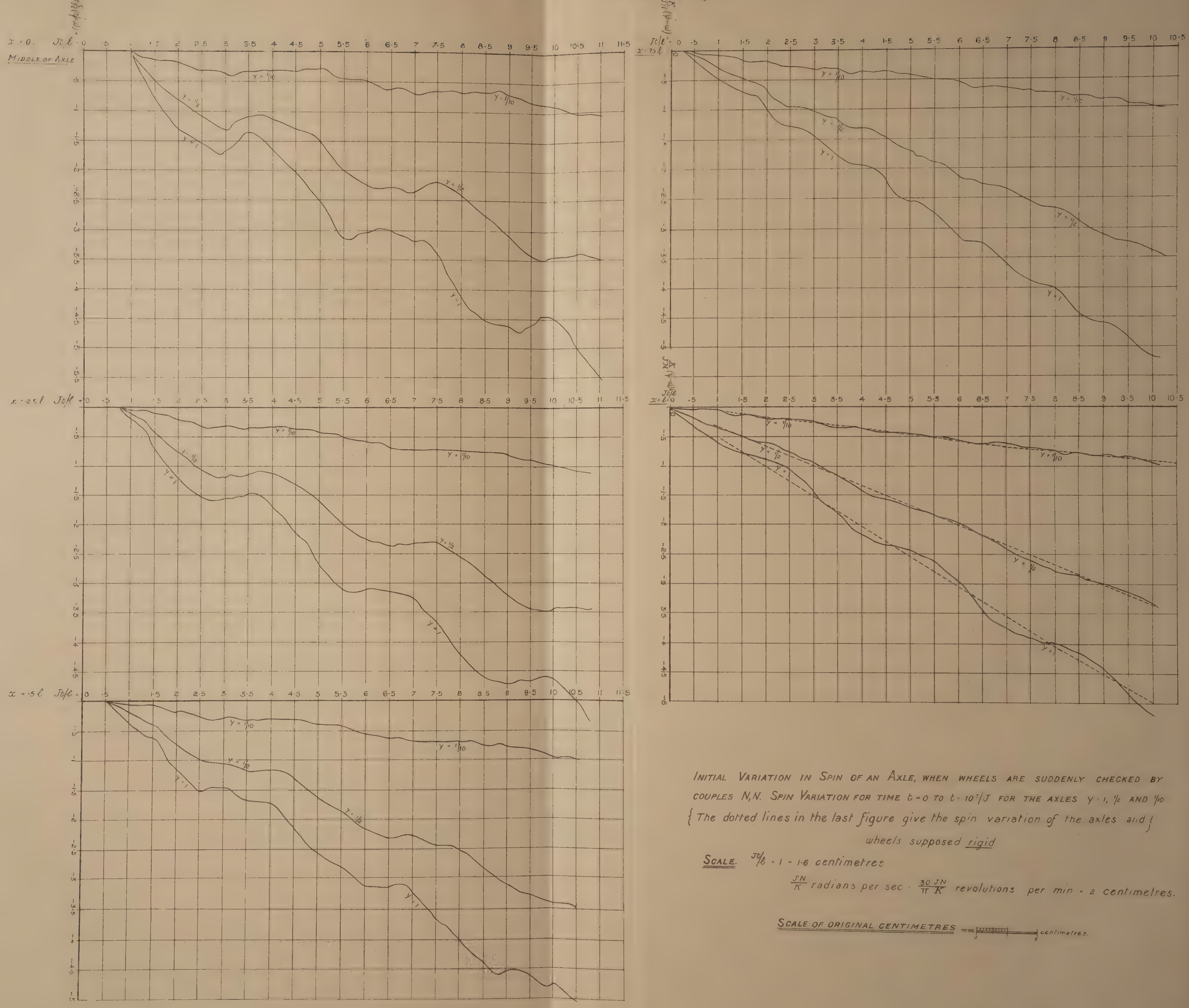


SCALE. $\xi/l = 1 = 1{\cdot}6$ centimetres.
$\phi'(\xi) = 1 = 2$ centimetres.

SCALE OF ORIGINAL CENTIMETRES mm 0 1 2 centimetres.

INITIAL STRESS IN AXLES. DIAGRAMS OF $\frac{\phi(\xi)}{y}$ AND $\phi'(\xi)$ FOR $y = 1, 1/2, 1/10$ & $1/100$
VALUES OF ξ/l FROM 0 TO 11

PLATE IX *Fig XIII.*



INITIAL VARIATION IN SPIN OF AN AXLE, WHEN WHEELS ARE SUDDENLY CHECKED BY COUPLES N,N. SPIN VARIATION FOR TIME $t = 0$ TO $t = 10^{?}/J$ FOR THE AXLES $y = 1, \frac{1}{2}$ AND $\frac{1}{10}$

{The dotted lines in the last figure give the spin variation of the axles and wheels supposed rigid}

SCALE. $Jt/l = 1 = 1{\cdot}6$ centimetres

$\frac{JN}{K}$ radians per sec $= \frac{30\,JN}{\pi K}$ revolutions per min $= 2$ centimetres.

SCALE OF ORIGINAL CENTIMETRES mm 0 1 2 centimetres.

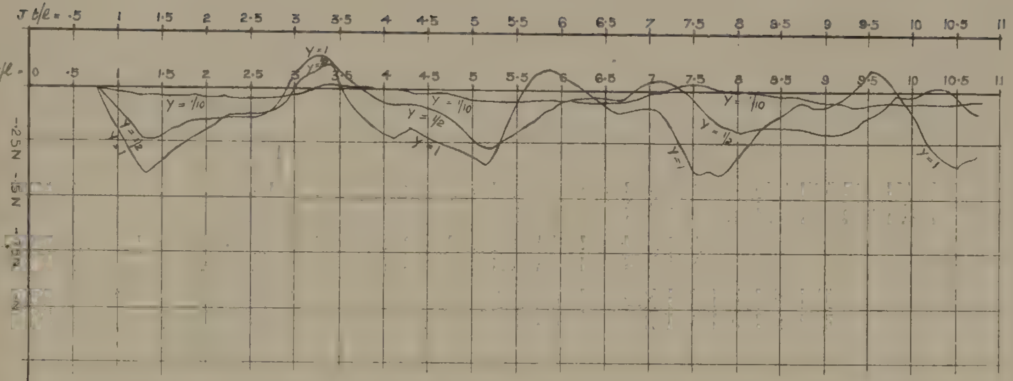


Fig. XIV(a).

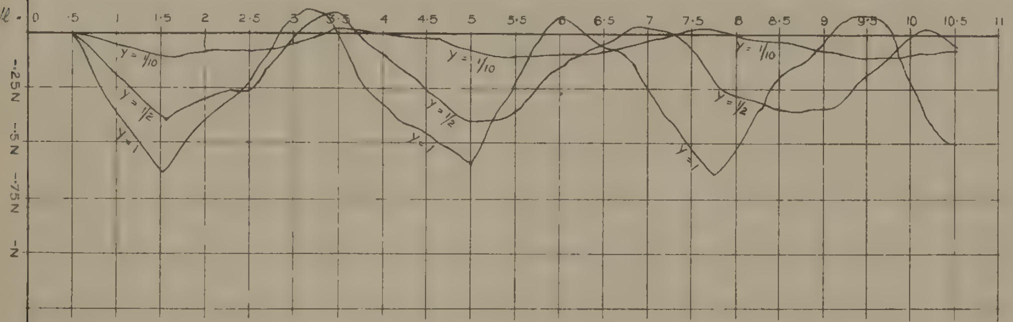


Fig. XIV(b).

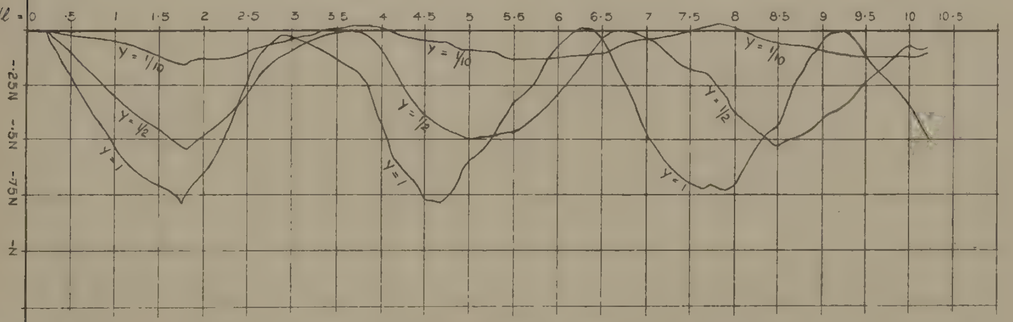


Fig. XIV(c).

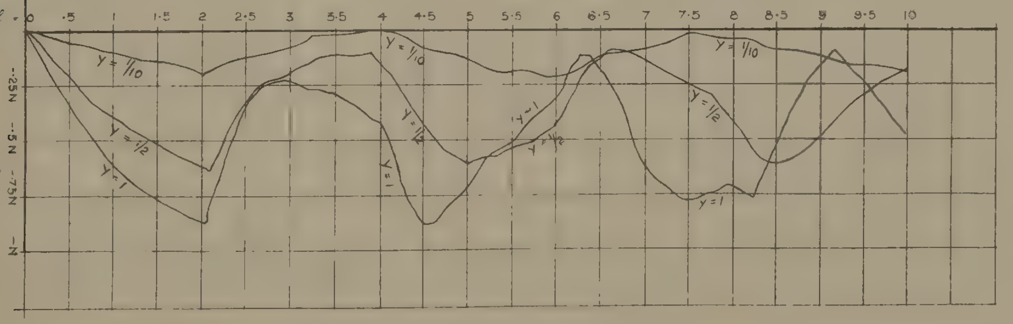


Fig. XIV(d).

INITIAL STRESS IN AXLE WHEN BRAKE IS SUDDENLY APPLIED.
VALUES OF TORSIONAL COUPLE $(K\frac{d\omega}{dz})$ FOR FIVE POINTS OF THE HALF-AXLE, WHEN $y = 1$, $\frac{1}{2}$ AND $\frac{1}{10}$.

SCALES: Unit of Jt/l = 1·6 centimetres.
Torsional Couple. N = 4 centimetres.

SCALE OF ORIGINAL CENTIMETRES mm. 0 5 centimetres.

Fig. XV.

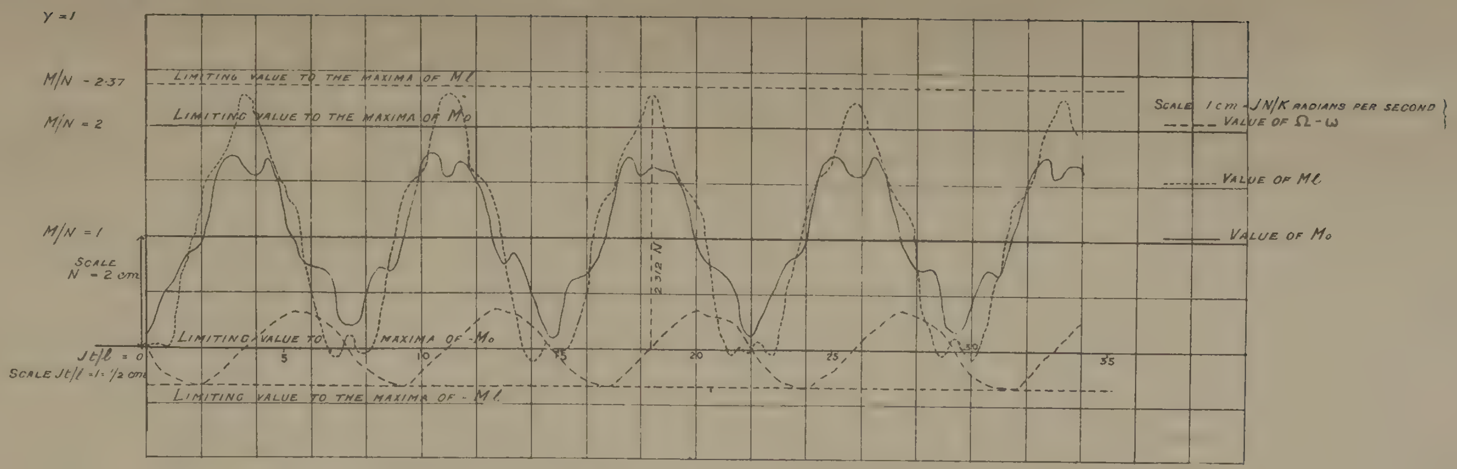
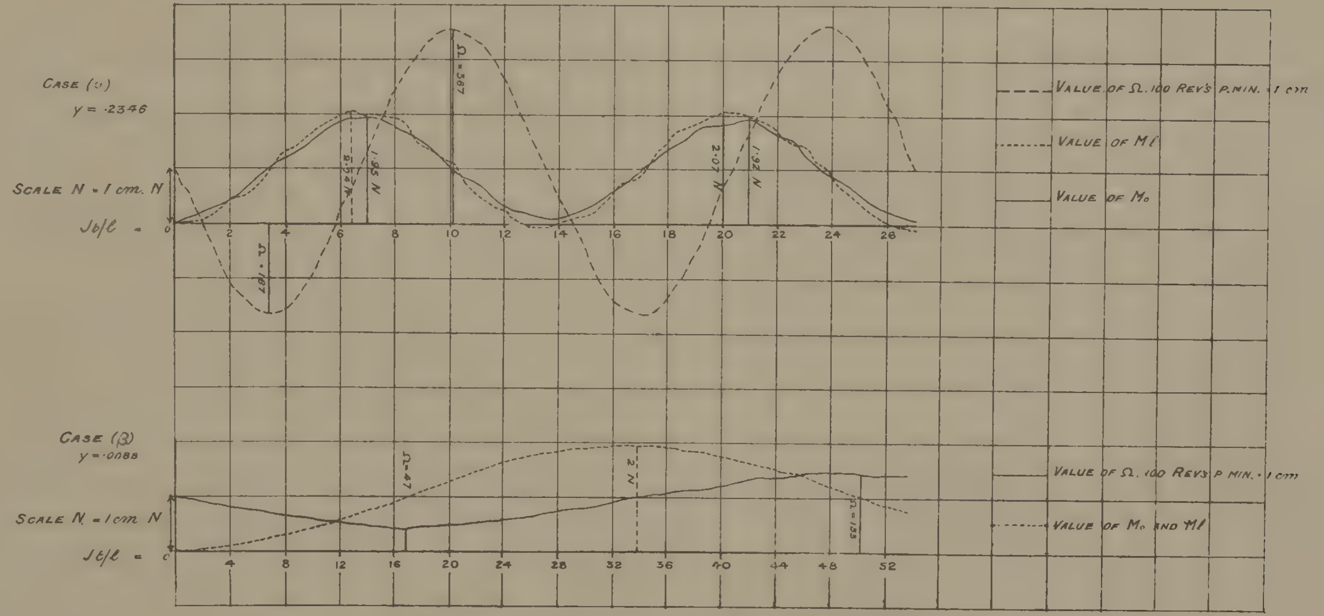


Fig. XVI.



SCALE OF ORIGINAL CENTIMETRES mm 0 1 2 centimetres.

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